

University of British Columbia

Math 301

Midterm 1

Date: February 6, 2016
Time: 11:00 - 11:50pm

Name (print):

Student ID Number:

Signature:

SOLUTIONS

Instructor: Richard Froese

Instructions:

1. No notes, books or calculators are allowed.
2. Read the questions carefully and make sure you provide all the information that is asked for in the question.
3. Show all your work. Answers without any explanation or without the correct accompanying work could receive no credit, even if they are correct.
4. Answer the questions in the space provided. Continue on the back of the page if necessary.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
Total:	40	

1. (a) (5 points) Find all the singular points of $\frac{\tan(\pi z)}{z}$ and classify them as essential, removable or a pole of a certain order. At each pole, find the residue.

Singular points are $z=0$ and points where $\cos(\pi z) = 0$,
i.e. $z = \frac{1}{2} + k, k \in \mathbb{Z}$.

At $z=0$, $\tan(\pi z) = \pi z + O(z^2)$ so $\frac{\tan(\pi z)}{z} = \pi + O(z)$

The singularity at 0 is removable

At $z = \frac{1}{2} + k$ write $\frac{\tan(\pi z)}{z} = \frac{p(z)}{q(z)}$ where $p(z) = \frac{\sin(\pi z)}{z}$

$q(z) = \cos(\pi z)$. Then $p(\frac{1}{2} + k) = \frac{\sin(\pi(\frac{1}{2} + k))}{\frac{1}{2} + k} = \frac{(-1)^k}{\frac{1}{2} + k} \neq 0$

$q'(\frac{1}{2} + k) = 0$, $q'(\frac{1}{2} + k) = -\pi \sin(\pi(\frac{1}{2} + k)) \neq 0$.

So $\frac{1}{2} + k$ is a simple pole with residue $\frac{\sin(\pi(\frac{1}{2} + k))}{(\frac{1}{2} + k)(-\pi) \sin(\pi(\frac{1}{2} + k))}$

- (b) (5 points) Compute

$$\oint_{|z|=1} \frac{\tan(\pi z)}{z} dz = \frac{-1}{\pi(\frac{1}{2} + k)}$$

where the contour is traversed once counterclockwise.

The contour includes $\frac{1}{2} + k$ for $k = -1, 0$, i.e. $z \in \{-\frac{1}{2}, \frac{1}{2}\}$.

$$\oint_{|z|=1} = 2\pi i \left(\text{Res}\left[-\frac{1}{2}\right] + \text{Res}\left[\frac{1}{2}\right] \right)$$

$$= 2\pi i \left(\frac{-1}{-(1/2)} + \frac{-1}{(1/2)} \right) = 0$$

2. Let

$$I_R = \oint_{|z|=R} \frac{z^2+1}{z^2-1} \sin(1/z) dz$$

(a) (5 points) Find I_2 (Hint: use the residue at infinity).

The Laurent series at infinity is

$$\frac{z^2+1}{z^2-1} \sin(1/z) = \left(\frac{1+\frac{1}{z^2}}{1-\frac{1}{z^2}} \right) \sin\left(\frac{1}{z}\right)$$

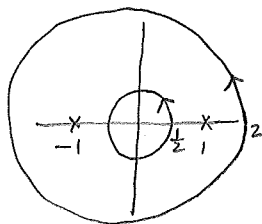
$$= \left(1 + \frac{1}{z^2}\right) \left(1 + \frac{1}{z^2} + \frac{1}{z^4} + \dots\right) \left(\frac{1}{z} - \frac{1}{3!} \frac{1}{z^3} + \dots\right)$$

$$= \frac{1}{z} + \frac{2}{z^3} + \dots \quad \text{So } \text{Res}[\infty] = 1 \text{ and}$$

$$I_2 = 2\pi i \quad \left[\text{Note: the standard defn for the residue at } \infty \text{ has an extra - sign. In this case, if } C \text{ is a very} \right.$$

large circle $\oint_C f(z) dz = -2\pi i \text{Res}[f; \infty]$ so I_2 is the same (Thanks J.L.)

(b) (5 points) Find $I_{1/2}$ (Hint: relate $I_{1/2}$ and I_2).



$$I_2 - I_{1/2} = 2\pi i (\text{Res}[1] + \text{Res}[-1])$$

$$I_{1/2} = I_2 - 2\pi i \left(\frac{1^2+1}{(1+1)} \sin(1) + \frac{(-1)^2+1}{-1-1} \sin(-1) \right)$$

$$= 2\pi i (1 - 2 \sin(1))$$

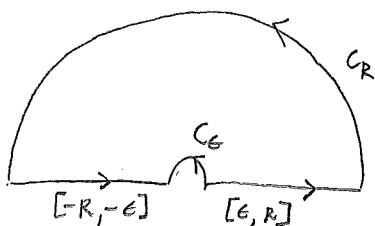
3. (10 points) Compute the integral

$$I = \int_0^{\infty} \frac{\cos(x) - 1}{x^2} dx$$

Write down the contours used to calculate it. Indicate the estimates you need to complete this calculation, but you need not prove them.

$$I = \frac{1}{2} \text{ p.v. } \int_{-\infty}^{\infty} \frac{\cos(x) - 1}{x^2} dx = \frac{1}{2} \text{ p.v. } \int_{-\infty}^{\infty} \frac{e^{ix} - 1}{x^2} dx$$

Let $f(z) = \frac{e^{iz} - 1}{z^2}$ and consider the contours



By C.I.F

$$\int_{[-R, -\epsilon]} f dz - \int_{C_\epsilon} f dz + \int_{[\epsilon, R]} f dz + \int_{C_R} f dz = 0$$

We have

$$\lim_{\epsilon \downarrow 0} \lim_{R \rightarrow \infty} \int_{[-R, -\epsilon]} f dz + \int_{[\epsilon, R]} f dz = 2I$$

$$\lim_{\epsilon \downarrow 0} \int_{C_\epsilon} f dz = i\pi \text{ Res}[f; 0]$$

$$\lim_{R \rightarrow \infty} \int_{C_R} f dz = 0$$

$$\text{Near } z=0 \quad f(z) = \frac{\left(1 + iz + \frac{1}{2!}(iz)^2 + \dots\right) - 1}{z^2} = \frac{i}{z} + \dots$$

So $\text{Res}[f; 0] = i$ and

$$2I - i\pi(i) + 0 = 0 \quad I = -\frac{\pi}{2}$$

4. (a) (3 points) Write down all the singular points and all the branch points of the multivalued function $\frac{z^\alpha}{1+z^4}$

If $\alpha \notin \mathbb{Z}$, then 0 and ∞ are branch pts. If $\alpha \in \mathbb{Z}$ there are no branch points.

Singular points are the points where $1+z^4=0$. These are $z = e^{i\pi/4}, e^{i3\pi/4}, e^{-i\pi/4}, e^{-i3\pi/4}$.

- (b) (7 points) Calculate the integral

$$I = \int_0^\infty \frac{x^\alpha}{1+x^4} dx$$

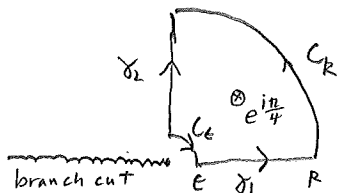
Specify the branch of any multivalued function you use and sketch the branch cut, along with the contours you are using. For what (real) values of α is your calculation valid? Give a reason.

Let z^α be the principal branch. This means that if we write $z = |z|e^{i\theta}$ with $\theta \in (-\pi, \pi]$ then

$z^\alpha = |z|^\alpha e^{i\alpha\theta}$. The branch cut is on $(-\infty, 0]$.

Let $f(z) = z^\alpha/(1+z^4)$.

We have $\int_{\gamma_1} + \int_{C_R} - \int_{\gamma_2} + \int_{C_\epsilon} = 2\pi i (\text{Res}[f; e^{i\pi/4}])$



$\int_{\gamma_1} f dz \rightarrow I$ as $\epsilon \downarrow 0, R \uparrow \infty$. $|\int_{C_\epsilon} f dz| \leq \frac{\epsilon^\alpha}{1-\epsilon^2} \frac{\pi}{2} \epsilon \rightarrow 0$ as

$\epsilon \rightarrow 0$ if $\alpha > -1$. $|\int_{C_R} f dz| \leq \frac{R^\alpha}{(R^4-1)} \frac{\pi}{2} R \rightarrow 0$ if $\alpha < 3$

To compute the integral over γ_2 parametrize with

$z(r) = e^{i\pi/2} r$. Then $z(r)^4 = r^4$, $z(r)^\alpha = r^\alpha e^{i\alpha\pi/2}$

$dz = e^{i\pi/2} dr$. So $\int_{\gamma_2} f dz \rightarrow e^{i\alpha\pi/2} e^{i\pi/2} I$

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$$\text{Res}[f; e^{i\pi/4}] = \frac{e^{i\pi\alpha/4}}{4(e^{i\pi/4})^3} = \frac{1}{4 e^{i\pi/4}(-\alpha+3\pi)}$$

We arrive at

$$I - e^{\frac{i\pi}{2}(\alpha+1)} I = \frac{2\pi i}{-4 e^{\frac{i\pi}{4}(-\alpha-1)}}$$

$$\begin{aligned} e^{\frac{3i\pi}{4}} &= -e^{\frac{-i4\pi}{4}} e^{\frac{3i\pi}{4}} \\ &= -e^{\frac{-i\pi}{4}} \end{aligned}$$

$$I = \frac{2\pi i}{(-4) e^{-i\frac{\pi}{4}(\alpha+1)} (1 - e^{\frac{i\pi}{2}(\alpha+1)})}$$

$$= \frac{2\pi i}{(-4) (e^{-\frac{i\pi}{4}(\alpha+1)} - e^{\frac{i\pi}{4}(\alpha+1)})}$$

$$= \frac{\pi}{4 \sin(\frac{\pi}{4}(\alpha+1))}$$

The estimates for $\int_{C_E}$ and $\int_{C_R}$ require $-1 < \alpha < 3$.