University of British Columbia
Math 301

Midterm 2

Date: March 22, 2016
Time: 11:00 - 11:50pm

Name (print):
Student ID Number:
Signature:

Instructor: Richard Froese

Instructions:

1. No notes, books or calculators are allowed.
2. Read the questions carefully and make sure you provide all the information that is asked for in the question.
3. Show all your work. Answers without any explanation or without the correct accompanying work could receive no credit, even if they are correct.
4. Answer the questions in the space provided. Continue on the back of the page if necessary.

Question	Points	Score
1	17	
2	10	
3	8	
4	5	
Total:	40	

1. Let R' be the region $0 < u < \frac{\pi}{2}$, $0 < v < \ln(2)$ in the $w = u + iv$ plane. The map

$$\cos(w) = \cos(u + iv) = \cos(u)\cosh(v) - i\sin(u)\sinh(v)$$

maps R' to the region R in the $z = x + iy$ plane shown on the right.

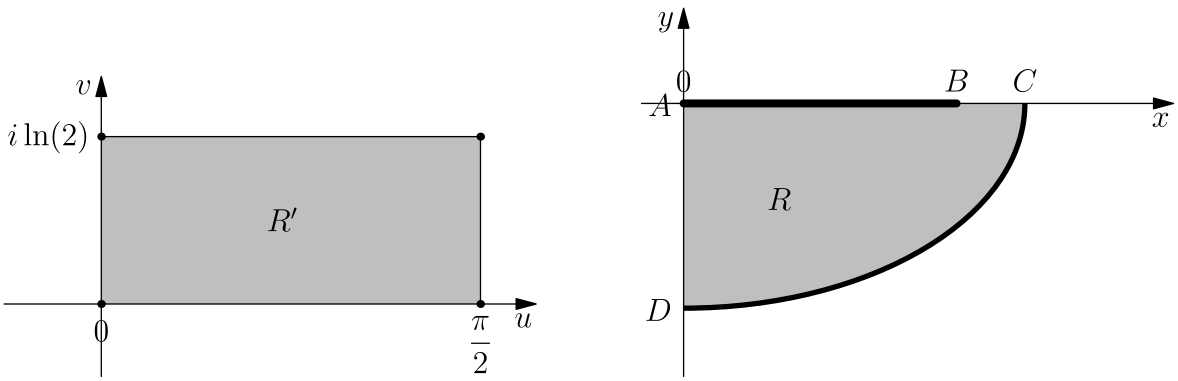


Figure 1: The regions R' and R

(a) (4 points) The corners of R' map to the points $A = 0$, B, C and D on the boundary of R. Find the values of the points B, C and D in the $z = x + iy$ plane. Indicate which corner maps to which of these points by labeling the corners of R' with A', B', C' or D'.

Solution: First compute the images of points on the left. We have $\cos(0) = 1$, $\cos(\pi/2) = 0$, $\cos(i\ln(2)) = \cosh(\ln(2)) = (e^{\ln(2)} + e^{-\ln(2)})/2 = (2 + 1/2)/2 = 5/4$, $\cos(\pi/2 + i\ln(2) = -i\sinh(\ln(2)) = -3i/4$. Examining these points we see that $A = 0$, $B = 1$, $C = 5/4$ and $D = -3i/4$ so that $A' = \pi/2$, $B' = 0$, $C' = i\ln(2)$ and $D' = \pi/2 + i\ln(2)$

(b) (5 points) Solve the equation $\cos(w) = (e^{iw} + e^{-iw})/2 = z$ for w to obtain a formula for the (multivalued) inverse map $w = \cos^{-1}(z)$. Let $f(z)$ be the the branch obtained by choosing the principal branches for the roots and logarithms in your formula. Verify that this choice of branch maps R to R' by showing that $f(A)$ ($= f(0)$) is one of the corners of R'.

Solution: The equation can be written $(e^{iw})^2 - 2ze^{iw} + 1 = 0$ so that $e^{iw} = z \pm (z^2 - 1)^{1/2}$ (where we choose the principal branch of the square root, and then $\pm$ gives both branches). So $w = \cos^{-1}(z) = -i\log(z \pm (z^2 - 1)^{1/2})$ as a multivalued function. Choosing principal branches gives $f(z) = -i\,\mathrm{Log}(z + (z^2 - 1)^{1/2})$. We know that $(-1)^{1/2} = i = e^{i\pi/2}$ (principal branch). So $f(A) = f(0) = -i\,\mathrm{Log}(e^{i\pi/2}) = -i^2\pi/2 = \pi/2 = A'$.

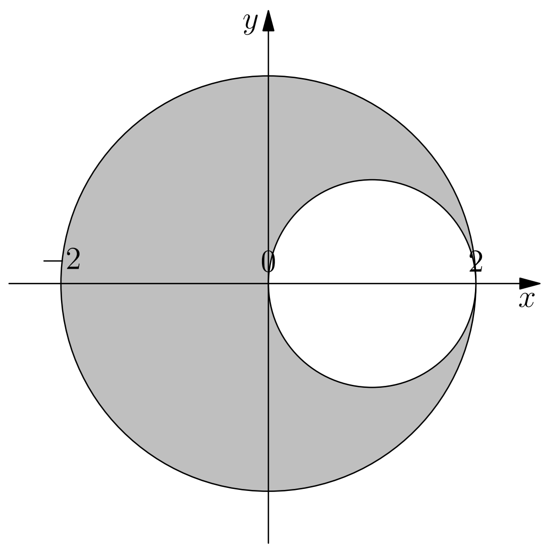
(c) (3 points) Suppose that R represents a plate that is insulated along the segment $[A, B]$ and the curve joining C to D, while the temperature is held at $0°$ on $[A, D]$ and at $100°$ on $[B, C]$. What boundary conditions are satisfied by the harmonic function $\phi(x, y)$ on R that represents the steady state temperature distribution?

Solution: The boundary conditions are $\phi = 0$ on $[A, D]$, $\phi = 100$ on $[B, C]$ and $\partial\phi/\partial n = 0$ on $[A, B]$ and on the boundary curve joining C to D.

(d) (5 points) We know that $\phi(x, y)$ corresponds to a harmonic function $\Phi(u, v)$ under the map $f(z)$. Indicate the boundary conditions satisfied by $\Phi(u, v)$ on the boundary of R'. Find $\Phi(u, v)$ and write the solution $\phi(x, y)$ in terms of Φ and $u(x, y)$, $v(x, y)$, where $f(x+iy) = u(x, y)+iv(x, y)$. **You need not calculate** $u(x, y)$ **and** $v(x, y)$.

Solution: We have $\Phi = 0$ on $[A', D']$, $\Phi = 100$ on $[B', C']$ and $\partial\phi/\partial n = \partial\phi/\partial u = 0$ on $[A', B']$ and $[C', D']$. The solution is $\Phi(u, v) = 100(1 - 2u/\pi)$ so that

$$\phi(x, y) = 100(1 - 2u(x, y)/\pi).$$



2. (a) (5 points) Find a fractional linear transformation that maps the shaded region between the circles to the strip $0 < \operatorname{Im} z < \pi$

Solution: A FLT that maps $2 \mapsto \infty$, $0 \mapsto 0$ and $1 + i \mapsto \alpha$ where $\alpha > 0$ is given by $z \mapsto i\alpha z/(z-2)$. Under this transformation, the inner circle maps to the real line, while the outer circle maps to a parallel line. We can vary which line by adjusting α. Since $-2 \mapsto i\alpha/2$ we must choose $\alpha = 2\pi$. Thus an FLT with the required properties is

$$z \mapsto \frac{2\pi i z}{z-2}.$$

If we follow this by a translation in a real direction we will get other FLT's that also work, namely

$$z \mapsto \frac{2\pi i z}{z-2} + a = \frac{(2\pi i + a)z - 2a}{z-2}$$

for any $a \in \mathbb{R}$.

(b) (5 points) Find a conformal map $f(z)$ that maps the shaded region between the circles to the upper half plane.

Solution: The exponential function maps the strip $0 < \operatorname{Im} z < \pi$ to the upper half plane. Therefore the function obtained by following one of the transformations in (a) with the exponential map, e.g.,

$$f(z) = e^{2\pi i z/(z-2)},$$

does the job.

3. (a) (5 points) Show that the complex velocity potential $\Omega(z) = v_0(z + a^2/z)$, where $v_0 > 0$ and $a > 0$, represents ideal fluid flow around the obstacle $|z| \leq a$. What is the velocity of the flow in the limit $|z| \to \infty$? Where are the stagnation points (where the velocity is zero)?

Solution: The complex velocity is given by $\overline{\Omega'(z)} = v_0(1 - \overline{a}^2/\overline{z}^2)$. So the limiting velocity is

$$\lim_{|z|\to\infty} v_0(1 - \overline{a}^2/\overline{z}^2) = v_0.$$

The stagnation points occur when $\overline{\Omega'(z)} = 0$. This happens when $z = \pm a$.

(b) (3 points) Let $\Omega(z)$ be as above and consider $\Omega(iz)$. Does this complex velocity potential also represent ideal fluid flow around the obstacle $|z| \leq a$? Give a reason.

Solution: We can write points on the boundary of the obstacle as $z = ae^{i\theta}$ for $\theta \in [0, 2\pi]$. For these points z we have

$$\Omega(iz) = v_0(iae^{i\theta} + a^2/(iae^{i\theta})) = v_0 ia(e^{i\theta} - e^{-i\theta}) = -2v_0 a \sin(\theta)$$

so that for these points z, $\mathrm{Im}(\Omega(iz)) = 0$. This shows that this complex velocity does represent ideal fluid flow around the obstacle.

4. (5 points) How many zeros does $p(z) = z^6 - 4z^4 + z^3 - i$ have inside the unit circle?

Solution: Let $f(z) = z^6 - 4z^4 = z^4(z^2 - 4)$. If we count with multiplicity, $f(z)$ has four zeros inside the unit circle. On the boundary where $|z| = 1$ we have $|f(z)| \geq 4|z|^4 - |z|^6 = 4 - 1 = 3$. On the other hand $g(z) = +z^3 - i$ satisfies $|g(z)| \leq |z|^3 + 1 = 2$ on the boundary. Thus $|g(z)| < |f(z)|$ on the boundary. Then Rouché's theorem implies tha t $f(z) + g(z) = p(z)$ also has 4 zeros inside the unit circle, counted with multiplicity.