

University of British Columbia

Math 301 Midterm 1

February 2, 2018 11:00 - 11:50am

Last Name (print):

First Name (print):

Student ID Number:

Signature:

Instructor: Richard Froese

Instructions:

1. No notes, books or calculators are allowed.
2. Read the questions carefully and make sure you provide all the information that is asked for in the question.
3. Show all your work. Answers without any explanation or without the correct accompanying work could receive no credit, even if they are correct.
4. Answer the questions in the space provided. Continue on the back of the page if necessary.

Question	Points	Score
1	8	
2	4	
3	16	
4	12	
Total:	40	

1. (a) (4 points) Evaluate

$$I = \int_0^{\infty} \frac{\cos(x)}{(x^2+1)} dx.$$

$$I = \frac{1}{2} \operatorname{Re} \int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+1} dx, \quad \text{Close contour in upper half}$$

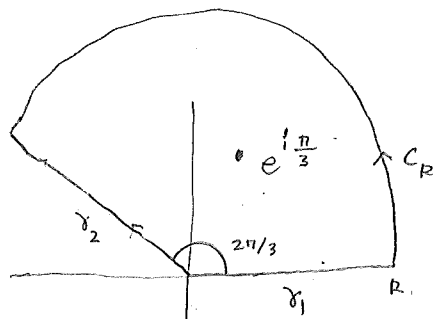
plane and use Jordan's lemma . This gives

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+1} dx = 2\pi i \operatorname{Res} \left[\frac{e^{iz}}{z^2+1}; i \right]$$

$$= 2\pi i \frac{e^{ii}}{2i} = \frac{\pi}{e}$$

$$I = \frac{\pi}{2e}$$

(b) (4 points) Evaluate



$$I = \int_0^{\infty} \frac{1}{(x^3+1)} dx.$$

$$\text{Let } f(z) = \frac{1}{z^3+1}. \quad \text{Then}$$

$$\begin{aligned} \int_{\gamma_1} f dz + \int_{C_R} f dz - \int_{\gamma_2} f dz &= 2\pi i \operatorname{Res} \left[\frac{1}{z^3+1}, e^{i\pi/3} \right] \\ &= 2\pi i \frac{1}{3(e^{i\pi/3})^2} = \frac{2\pi i}{3e^{i2\pi/3}} \end{aligned}$$

Now $\int_{\gamma_1} f dz \rightarrow I$ as $R \rightarrow \infty$,

$\int_{C_R} f dz \rightarrow 0$ since $|\text{integrand}| < \frac{C}{R^3}$ and $\text{length}(C_R) = CR$.

We can parametrize γ_2 with $z(r) = e^{2i\pi/3} \cdot r$, $r \in [0, R]$. Then $z'(r) = e^{2i\pi/3}$

$$\text{so } \int_{\gamma_2} f dz = \int_0^R \frac{1}{(e^{2i\pi/3} r)^3 + 1} e^{2i\pi/3} dr \Rightarrow e^{2i\pi/3} \cdot I \text{ as } R \rightarrow \infty.$$

$$\text{Thus } (1 - e^{2i\pi/3}) I = \frac{2\pi i}{3e^{i2\pi/3}}, \quad I = \frac{2\pi i}{3(e^{i2\pi/3} - e^{i4\pi/3})}$$

$$= \frac{2\pi i}{3(e^{2i\pi/3} - e^{-2i\pi/3})} = \frac{\pi}{3 \sin(\frac{2\pi}{3})} \quad \text{Page 2} \quad \left(\text{also} = \frac{2\pi\sqrt{3}}{9} \right).$$

2. (4 points) If $f(z)$ is analytic with a zero of order n at $z = z_0$, what kind of singularity does $f'(z)/f(z)$ have at $z = z_0$? What is the residue?

We have $f(z) = (z - z_0)^n h(z)$ where $h(z)$ is analytic and $h(z_0) \neq 0$. Then $f'(z) = n(z - z_0)^{n-1} h(z) + (z - z_0)^n h'(z)$

So

$$\frac{f'}{f} = \frac{n}{(z - z_0)} + \frac{h'(z)}{h(z)}$$

Since $h(z_0) \neq 0$, h'/h is analytic near z_0 . So

f'/f has a simple pole with residue n .

3. (a) (4 points) Where are the branch points of $(z^2 + 1)^{1/2}$? Is infinity a branch point?

Since $z^2 + 1 = (z+i)(z-i)$ there are branch points at i and $-i$. Explicitly $(z^2 + 1)^{\frac{1}{2}} = (z+i)^{\frac{1}{2}}(z-i)^{\frac{1}{2}}$ where we choose a branch for each factor in the right to get a branch of the original function on the left. When we move z in a small circle around i , $(z-i)^{\frac{1}{2}} \rightarrow (-1)(z-i)^{\frac{1}{2}}$ while $(z+i)^{\frac{1}{2}}$ returns to the same value. Thus $(z^2 + 1)^{\frac{1}{2}} \rightarrow (-1)(z^2 + 1)^{\frac{1}{2}}$ so i is a branch pt. Similarly, $-i$ is a branch pt. To check ∞ we can write $(z^2 + 1)^{\frac{1}{2}} = z(1 + \frac{1}{z^2})^{\frac{1}{2}}$. This returns to the same value when z goes around a large circle. Thus ∞ is not a branch pt.

(b) (4 points) Using the range of angles method, construct a branch of $(z^2 + 1)^{1/2}$ that

- has branch cut on the imaginary axis,
- is analytic outside the unit circle, and
- is positive on the real axis.

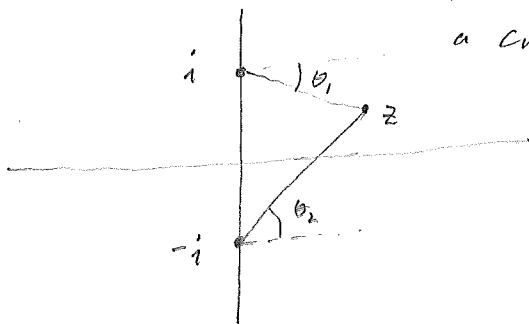
The range of angles method defines a branch of $(z^2 + 1)^{\frac{1}{2}}$ as $|z^2 + 1|^{\frac{1}{2}} e^{i(\frac{\theta_1 + \theta_2}{2})}$ where $(z_j - i) = |z_j - i| e^{i\frac{\theta_j}{2}}$ $j = 1, 2$.

and $\theta_1 \in I_1$, $\theta_2 \in I_2$ for some choice of intervals I_1, I_2

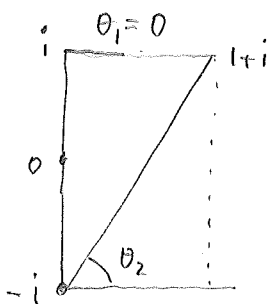
of length 2π . If we choose $I_1 = I_2 = [-\frac{\pi}{2}, \frac{3\pi}{2})$ (or $(-\frac{\pi}{2}, \frac{3\pi}{2}]$)

the cuts on the negative imag axis will cancel, leaving a cut on $[-i, i]$. On the positive real axis

$\theta_1 = -\theta_2$ so our branch is $|z^2 + 1|^{\frac{1}{2}} > 0$ there.



(c) (4 points) What is the value of the branch constructed in (b) at $z = 1 + i$?



$$\begin{aligned}
 \text{When } z = 1+i \text{ then } \theta_1 = 0, \theta_2 = \tan^{-1}(2) \\
 \text{branch} &= |(1+i)^2 + 1|^{\frac{1}{2}} e^{i \frac{\tan^{-1}(2)}{2}} \\
 &= |2i + 1|^{\frac{1}{2}} e^{i \frac{\tan^{-1}(2)}{2}} \\
 &= 5^{\frac{1}{4}} e^{i \frac{\tan^{-1}(2)}{2}}
 \end{aligned}$$

(d) (4 points) What is

$$\oint_{|z|=2} \frac{(z^2 + 1)^{1/2}}{z^2}$$

(traversed once counterclockwise). Here $(z^2 + 1)^{1/2}$ is the branch constructed above.
(Hint: compute the residue at infinity).

$$\text{Since } \left| \frac{(z^2 + 1)^{\frac{1}{2}}}{z^2} \right| = \frac{|z^2 + 1|^{\frac{1}{2}}}{|z|^2} \rightarrow 0 \text{ as } |z| \rightarrow \infty,$$

$$\text{the residue at } \infty \text{ is } \lim_{x \rightarrow \infty} x \frac{(x^2 + 1)^{\frac{1}{2}}}{x^2} = 1$$

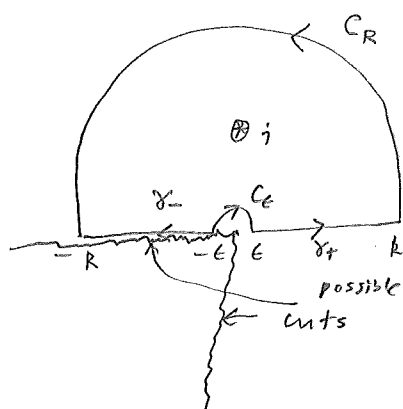
So the integral is $2\pi i$

4. This question is about evaluating

$$I = \int_0^\infty \frac{\ln(x)}{(x^2 + 1)^2} \quad \text{and} \quad J = \int_0^\infty \frac{1}{(x^2 + 1)^2}.$$

by choosing a branch Log_* of the log, integrating $f(z) = \text{Log}_*(z)/(x^2 + 1)^2$ around a suitable contour and taking limits.

- (a) (4 points) Indicate which branch of the log you would choose. Draw the contour, the branch cut of your choice and the positions of any singularities of $f(z)$ inside the contour.



We can use any branch with a cut in the lower half plane. We can also use the principle branch if we integrate along the top of the cut.

(b) (4 points) Evaluate the integral of f around the contour in part (a) using residues.

$$\text{Let } f(z) = \text{Log}(z) / (z^2+1)^2$$

$$\int_{\gamma_4} f dz + \int_{C_R} f dz - \int_{\gamma_-} f dz + \int_{C_\epsilon} f dz = 2\pi i \text{Res}[f; i]$$

$$\text{Res}\left[\frac{\text{Log}(z)}{(z^2+1)^2}; i\right] = \text{Res}\left[\frac{\text{Log}(z)}{(z+i)^2(z-i)^2}; i\right]$$

$$= \lim_{z \rightarrow i} \frac{d}{dz} \frac{\text{Log}(z)}{(z+i)^2} = \lim_{z \rightarrow i} \frac{\frac{1}{z}(z+i)^2 - \text{Log}(z) \cdot 2(z+i)}{(z+i)^4}$$

$$= \frac{-i(2i)^2 - i\pi/2 \cdot 2 \cdot 2i}{(2i)^4} = \frac{4i + 2\pi}{16}$$

$$2\pi i \cdot \text{Res} = -\frac{8\pi + 4\pi^2 i}{16} = -\frac{\pi}{2} + \frac{\pi^2 i}{4}$$

(c) (4 points) Take a limit to obtain an equation for I and J . Indicate which contour integrals are tending to zero in the limit. Solve for I and J .

$$\lim_{\epsilon \downarrow 0, R \uparrow \infty} \int_{\gamma_4} f dz = I$$

$$\lim_{C_R} \int f dz = 0 \quad \lim_{C_\epsilon} \int f dz = 0$$

For γ_- parametrize with $z(r) = -r$ $dz = -dr$ $r \in [\epsilon, R]$.

$$\int_{\gamma_-} f dz = \int_{\epsilon}^R \frac{\text{Log}(-r)}{((r)^2+1)^2} (-dr) = -\int_{\epsilon}^R \frac{\ln r + i\pi}{(r^2+1)^2} dr = -I - i\pi J$$

$$I + 0 - (-I - i\pi J) + 0 = -\frac{\pi}{2} + \frac{\pi^2 i}{4}$$

$$2I + i\pi J = -\frac{\pi}{2} + \frac{\pi^2 i}{4} \implies I = -\frac{\pi}{4} \quad J = \frac{\pi}{4}$$