University of British Columbia
Math 301 Midterm 1
February 7, 2020 11:00 - 11:50am

Last Name (print):

First Name (print):

Student ID Number:

Signature:

Instructor: Richard Froese

Instructions:

1. No notes, books or calculators are allowed.
2. Read the questions carefully and make sure you provide all the information that is asked for in the question.
3. Show all your work. Answers without any explanation or without the correct accompanying work could receive no credit, even if they are correct.
4. Answer the questions in the space provided. Continue on the back of the page if necessary.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
Total:	40	

1.

In this question, all functions are analytic except for the indicated singularities and branch cuts. Note: there are many correct answers to each question.

(a) (2 points) Write down an example of a function $f(z)$ with

- simple pole at z=0 and residue $\text{Res}[f; z = 0] = \pi$, and
- a pole of order 2 at z=1 and residue $\text{Res}[f; z = 1] = 0$

Solution: A possibility is

$$f(z) = \frac{\pi}{z} + \frac{1}{(z-1)^2}$$

The point is that $\dfrac{1}{(z-1)^2}$ is analytic near $z = 0$ so it's Laurent series at $z = 0$ is a Taylor series which does not interfere with the pole. Similarly for $\dfrac{\pi}{z}$ near $z = 1$.

(b) (2 points) Write down an example of a function $f(z)$ with

- a simple pole at every integer k and residue $\text{Res}[f, z = k] = 1$ for every $k \in \mathbb{Z}$.

Solution: Starting with $\sin(\pi z)$ which has a zero when $z = k \in \mathbb{Z}$ we could try $\dfrac{1}{\sin(\pi z)}$. But this has residue $\dfrac{1}{\pi \cos(\pi k)} = (-1)^k/\pi$ at $z = k \in \mathbb{Z}$. So let's multiply by an entire function whose value at integer points k is $\pi(-1)^k$. Such a function is $\pi \cos(\pi k)$. So

$$f(z) = \frac{\pi \cos(\pi z)}{\sin(\pi z)}$$

works.

(c) (2 points) Write down an example of a function $f(z)$ with

- an essential singularity at $z = 0$ and residue $\text{Res}[f, z = 0] = 5$.

> **Solution:** An example is $5e^{1/z}$, since this has Laurent series $5(1 + \frac{1}{z} + \frac{1}{2z^2} + \cdots) = 5 + \frac{5}{z} + \frac{5}{2z^2} + \cdots$.

(d) (2 points) Write down an example of a function $f(z)$ whose residue at infinity is defined and equal to zero, i.e., $\text{Res}[f, z = \infty] = 0$

> **Solution:** A function of the form $f(z) = \frac{1}{p(z)}$ where p is a polynomial of degree 2 or higher will work because for such function $\lim_{|z|\to\infty} f(z) = 0$ so the residue at infinity is $\lim_{|z|\to\infty} zf(z)$ which is also 0.
> (Some of you pointed out that $f(z) = 0$ also works! Well, a better question should ask for non-zero f.)

(e) (2 points) Write down an example of a multivalued function $f(z)$ with branch points at $z = 1$, $z = i$ and $z = -i$.

> **Solution:** An example is $f(z) = \log(z - 1) + \log(z - i) + \log(z + i)$

2. (a) (6 points) Compute the integral

$$\int_{-\infty}^{\infty} \frac{e^{2ix}}{x^2+9} dx$$

Solution: We close the contour in the upper half plane. There is only one singularity at $z = 3i$. So the integral is

$$2\pi i \operatorname{Res}\left[\frac{e^{2iz}}{z^2+9}, z = 3i\right] = 2\pi i \frac{e^{2i(3i)}}{2 \cdot 3i} = \frac{\pi e^{-6}}{3}$$

and use your answer to compute

$$\int_{-\infty}^{\infty} \frac{\cos(2x)}{x^2+9} dx =$$

Solution: $\dfrac{e^{-6}}{3\pi}$ (take the real part)

$$\int_{-\infty}^{\infty} \frac{\sin(2x)}{x^2+9} dx =$$

Solution: 0 (take the imaginary part)

$$\int_{-\infty}^{\infty} \frac{e^{-2ix}}{x^2+9} dx =$$

Solution: $\dfrac{e^{-6}}{3\pi}$ (take the complex conjugate)

(b) (4 points) Compute the integral

$$\int_{-\infty}^{\infty} \frac{1}{x^2+i} dx$$

Solution: We can close the contour in either half plane. The singularities of $\frac{1}{z^2+i}$ are solutions of $z^2 = -i = e^{-i\pi/2}$ There are two, namely, $z = \pm e^{-i\pi/4}$. The singularity in the upper half plane is $z = -e^{-i\pi/4} = e^{3i\pi/4}$ This leads to

$$\int_{-\infty}^{\infty} \frac{1}{x^2+i} dx = 2\pi i \frac{1}{2e^{3i\pi/4}} = \pi e^{-i\pi/4} = \frac{\pi(1-i)}{\sqrt{2}}$$

3. (10 points) Using the range of angles method, construct a branch of $(1-z^2)^{1/2}$ that

- has a branch cut on $[-1,1]$, and
- takes on *negative* values on the cut when approached from above.

Your answer should contain explicit instructions on how to compute the value of your branch at z for every $z \in \mathbb{C}$ and a sketch showing the angles.

Solution: To implement the range of angles method start with the factorization $(1-z^2) = (-1)(z-1)(z+1)$ and write each factor in polar form. This gives

$$(1-z^2) = e^{\pm i\pi}|z-1|e^{i\theta_1}|z+1|)e^{i\theta_2} = |1-z^2|e^{i(\pm\pi+\theta_1+\theta_2)}$$

Then, for every choice of sign for $\pm\pi$, and every choice of $\theta_j \in \arg(z-z_j)$, $j=1,2$ the quantity

$$|1-z^2|^{1/2}e^{i\frac{1}{2}(\pm\pi+\theta_1+\theta_2)}$$

is one of the multiple values of $(1-z^2)^{1/2}$.

The range of angles method is a consistent way of choosing the angles θ_j above. Fix an interval I_j of length 2π for each angle θ_i. There is always exactly one $\theta_j \in \arg(z-z_j) \cap I_j$. This is the angle we choose

So, for this question let's choose $\pm\pi = \pi$ and the intervals $I_1 = I_2 = [0, 2\pi)$.

The angle θ_1 will jump by 2π when z crosses $[1,\infty)$ on the real axis.

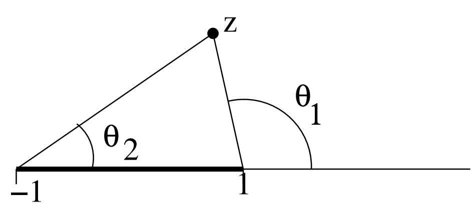
The angle θ_2 will jump by 2π when z crosses $[-1,\infty)$ on the real axis.

On the portion $[1,\infty)$ the total jump in $\theta_1+\theta_2$ is 4π. Since we are dividing by 2 this means that the branch cuts cancel on $[1,\infty)$ and we are left with a branch cut on $[-1,1]$. When $z \downarrow x \in [-1,1]$, $\theta_1 \to \pi$ and $\theta_2 \to 0$.

So

$$|1-z^2|^{1/2}e^{i\frac{1}{2}(\pm pi+\theta_1+\theta_2)} \to |1-z^2|^{1/2}e^{i\frac{1}{2}(\pi+\pi+0)} = e^{i\pi}|1-z^2|^{1/2} = (-1)|1-z^2|^{1/2} \le 0$$

Explicit instructions: Given z, let $\theta_k = \theta_k(z)$ be the unique element in $\arg(z-z_k) \cap [0,2\pi)$. Then our branch evaluated at z is $|1-z^2|^{1/2}e^{i\frac{1}{2}(\pi+\theta_1+\theta_2)}$.



4. (10 points) Compute

$$\int_0^\infty \frac{\sin(x)}{x} dx.$$

Explain each of your steps (e.g., why you must introduce a principal value integral and an indented contour) and sketch the contours that you use.

Solution: We integrate $f(z) = \dfrac{e^{iz}}{z}$ around the closed contour $[-R, -\epsilon] - C_\epsilon + [\epsilon, R] + C_R$. Here C_ϵ and C_R are semi-circle contours in the the upper half plane, traversed counter-clockwise. Since the singularity at $z = 0$ is outside the contour, Cauchy's theorem says

$$\int_{-R}^{-\epsilon} f(z)dz + \int_{\epsilon}^{R} f(z)dz = \int_{C_\epsilon} f(z)dz - \int_{C_R} f(z)dz.$$

Since $f(z)$ has a simple pole at $z = 0$, we know

$$\lim_{\epsilon\to 0} \int_{C_\epsilon} f(z)dz = i\pi \operatorname{Res}[f, z = 0] = i\pi.$$

Thus

$$\lim_{\epsilon\to 0} \left[\int_{-R}^{-\epsilon} f(z)dz + \int_{\epsilon}^{R} f(z)dz\right] = i\pi - \int_{C_R} f(z)dz.$$

By Jordan's lemma, we have

$$\lim_{R\to\infty} \int_{C_R} f(z)dz = 0.$$

Thus

$$\lim_{R\to\infty} \lim_{\epsilon\to 0} \left[\int_{-R}^{-\epsilon} f(z)dz + \int_{\epsilon}^{R} f(z)dz\right] = i\pi$$

and therefore

$$\operatorname{Im}\left(\lim_{R\to\infty} \lim_{\epsilon\to 0} \left[\int_{-R}^{-\epsilon} f(z)dz + \int_{\epsilon}^{R} f(z)dz\right]\right) = \operatorname{Im}(z)(i\pi) = \pi.$$

Now we take the imaginary part. Since $z \mapsto \operatorname{Im}(z)$ is continuous we can exchange taking $\operatorname{Im}(z)(i\pi)$ with the limit. For $z = x \in \mathbb{R}$, $\operatorname{Im} f(x) = \dfrac{\sin x}{x}$, so this yields

$$\lim_{R\to\infty} \lim_{\epsilon\to 0} \left[\int_{-R}^{-\epsilon} \frac{\sin x}{x} dx + \int_{\epsilon}^{R} \frac{\sin x}{x} dz\right] = \pi$$

The function $\dfrac{\sin x}{x}$ has a removable singularity at $x = 0$. Therefore it is continuous on $[-R, R]$, its integral over $[-R, R]$ exists and

$$\lim_{\epsilon\to 0} \left[\int_{-R}^{-\epsilon} \frac{\sin x}{x} dx + \int_{\epsilon}^{R} \frac{\sin x}{x} dx\right] = \int_{-R}^{R} \frac{\sin x}{x} dx$$

So

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{\sin x}{x} dx = \pi$$

Finally we note that $\sin(x)/(x)$ is even. This implies

$$\int_{0}^{\infty} \frac{\sin x}{x} dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$$