Math 301 Final Exam

Apr 24, 2008

Duration: 150 minutes

Name: ____________________ Student Number: ____________________

1. **Do not open this test until instructed to do so!**
2. **Please place your student ID (or another picture ID) on the desk.**
3. This exam should have 4 pages, including this cover sheet.
4. No textbooks, calculators, or other aids are allowed.
5. One page of notes (two-sided) is allowed.
6. Turn off any cell phones, pagers, etc. that could make noise during the exam.
7. **Circle your solutions! Reduce your answer as much as possible. Explain your work.**

Read these UBC rules governing examinations:

(i) Each candidate must be prepared to produce, upon request, a Library/AMS card for identification.

(ii) Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.

(iii) No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.

(iv) Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.

- Having at the place of writing any books, papers or memoranda, calculators, computers, audio or video cassette players or other memory aid devices, other than those authorized by the examiners.
- Speaking or communicating with other candidates.
- Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness shall not be received.

(v) Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.

1. (15 points) Evaluate the principal value integrals.

$$\text{(a)} \qquad I_1 = \int_{-\infty}^{\infty} \frac{e^{2ix}}{x-i} dx$$

$$\text{(b)} \qquad I_2 = \int_{-\infty}^{\infty} \frac{\cos 2x}{x-i} dx$$

2. (15 points) We define a *fixed point* of a function $f(z)$ to be any z_0 for which $f(z_0) = z_0$ and recall that a Möbius transformation is a function $M(z)$ of the form

$$M(z) = \frac{az+b}{cz+d}$$

subject to the requirement $ad \neq bc$.

(a) Find conditions on a, b, c, d that guarantee that every point $z \in \mathbb{C}$ is a fixed point of $M(z)$.

(b) Construct a Möbius transformation $M(z)$ with $c \neq 0$ that has a fixed point at $z_0 = 1 + i$, and no other fixed points.

(c) Assuming $c \neq 0$, specify the minimum and maximum numbers of fixed points that $M(z)$ can have. (Justify your answer.) Interpret your answer geometrically.

3. (15 points) Let D be the part of the complex plane satisfying $\mathrm{Im}(z) > 0$ and $|z - i| < a$ where $0 < a < 1$.

(a) Find a conformal map of D onto a region bounded by concentric circles centered at the origin.

(b) Find a solution of Laplace's equation $\nabla^2\phi = 0$ on the region D subject to the boundary condition that $\phi = 1$ on the circle $|z - i| = a$ and $\phi = 0$ on the line $\mathrm{Im}(z) = 0$.

(c) Now consider the case $a = 1$ so the circle touches the real axis in the z-plane. Verify that the corresponding solution of Laplace's equation is given by

$$\phi(x, y) = 1 - \mathrm{Re}\left[\frac{z - 2i}{z}\right] = \frac{2y}{x^2 + y^2}$$

where $z = x + iy$.

(d) Is it possible to obtain the formula of part (c) by taking the limit $a \to 1$ in the solution of part (b)? Explain why or why not on physical grounds.

4. (15 points) Consider the following initial value problem:

$$\begin{aligned} & y^{(4)} + ky''' + y'' + y' = e^{-t}, \quad k \geq 0 \\ \text{with} \qquad & y(0) = 1, y'(0) = y''(0) = y'''(0) = 0. \end{aligned}$$

(a) Calculate $Y(s)$, the Laplace transform of the solution $y(t)$.

(b) Let $k = 2$. Prove that $y(t)$ is bounded. Hint: use Nyquist criterion/argument principle.

5. (10 points) Use the Fourier transform to solve the diffusion equation:

$$\begin{aligned} u_t &= Du_{xx} \qquad -\infty < x < \infty, \quad t > 0 \\ u(x,0) &= f(x) \\ u(x,t) &\to 0 \ \text{ as } |x| \to \infty. \end{aligned}$$

You can freely use the following result for the Fourier Transform of a Gaussian function:

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/\sigma^2} \\ \hat{f}(k) &= e^{-\sigma^2 k^2/2} \end{aligned}$$

6. (10 points) Let $P(z)$ and $Q(z)$ be two polynomials such that $\deg(Q) \geq 2 + \deg(P)$. Suppose $\{z_1, \ldots, z_n\}$ is the set of all distinct poles of $P(z)/Q(z)$.

(i) Show that

$$\sum_{j=1}^{n} \text{Res}\{P(z)/Q(z); z_j\} = 0.$$

(ii) What is $\text{Res}\{P(z)/Q(z); \infty\}$?

7. (10 Points) Calculate the Fourier transform of

$$f(t) = \begin{cases} e^{-2t}\cos(t) & t \geq 0 \\ 0 & t < 0. \end{cases}$$

Hint: Use the connection between Fourier and Laplace transforms.

8. (10 points) (a) Explain briefly how we can use contour integration to calculate

$$\sum_{k=1}^{\infty} \frac{1}{k^2}.$$

(Provide a sketch of necessary steps.)

(b) Explain why the same technique cannot be used to evaluate

$$\sum_{k=1}^{\infty} \frac{1}{k^3}.$$