Basic Estimates

filename: `basicestimates.tex`

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Triangle inequality and variants

Basic triangle:

$$|z+w| \leq |z| + |w|.$$

Reverse triangle:

$$|z+w| \geq \Big| |z| - |w| \Big|.$$

Notice that $|z| - |w|$ is either equal to $\Big| |z| - |w| \Big|$ or negative. So it is also correct to write $|z+w| \geq |z| - |w|$ (and similarly $|z+w| \geq |w| - |z|$).

Generalized triangle:

$$\left| \sum_{i=1}^{n} z_i \right| \leq \sum_{i=1}^{n} |z_i|.$$

Generalized reverse triangle:

$$\left| \sum_{i=1}^{n} z_i \right| \geq |z_1| - \sum_{i=2}^{n} |z_i|$$

Integrals

By applying the generalized triangle inequality to the Riemann sum defining a complex contour integral and taking limits we find

$$\left| \int_{\Gamma} f(z) dz \right| \leq \int_{a}^{b} |f(z(t))| \, |z'(t)| dt.$$

Here $z(t)$, $t \in [a,b]$ is a parametrization of the contour Γ. This can be estimated further to yield

$$\left| \int_{\Gamma} f(z) dz \right| \leq \max_{z \in \Gamma} |f(z)| \cdot \text{length}(\Gamma)$$

Growth of polynomials and rational functions

Let $p(z)$ be a polynomial of degree n. This means $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ with $a_n \neq 0$. Let c, C be constants with $c < |a_n| < C$. Then for sufficiently large $|z|$ (i.e., there exists an $R > 0$ such that for all z with $|z| > R$)

$$c|z|^n \leq |p(z)| \leq C|z|^n$$

Let $p(z)$ be a polynomial of degree n with leading coefficient a_n and $q(z)$ be a polynomial of degree m with leading coefficient b_m. Let c, C be constants with with $c < |a_n|/|b_m| < C$ Then for sufficiently large $|z|$

$$c|z|^{n-m} \leq \left|\frac{p(z)}{q(z)}\right| \leq C|z|^{n-m}.$$