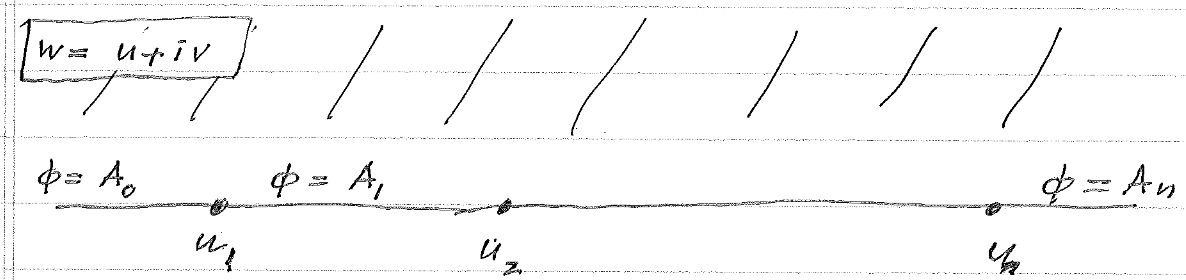
(1)

Solution of Dirichlet problem on U.H.P. with piecewise const b.c.



$$\Phi(u,v) = c_0 + c_1 \operatorname{Arg}(w-u_1) + \cdots + c_n \operatorname{Arg}(w-u_n)$$

where $w = u+iv$.

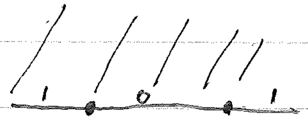
B.C. are satisfied if

$$\begin{aligned} c_0 + \pi c_1 + \cdots + \pi c_n &= A_0 \\ c_0 + 0 c_1 + \pi c_2 \cdots \pi c_n &= A_1 \\ &\vdots \\ c_0 + 0 c_1 + \cdots + 0 c_n &= A_n \end{aligned}$$

$$\begin{bmatrix} 1 & \pi & \pi & & \pi \\ 1 & 0 & \pi & & \vdots \\ \vdots & \vdots & 0 & & \\ & & \vdots & & \\ 1 & 0 & 0 & & 0 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} A_0 \\ \vdots \\ A_n \end{bmatrix} \qquad \begin{bmatrix} c_0 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} A_n \\ (A_0 - A_1)/\pi \\ \vdots \\ (A_{n-1} - A_n)/\pi \end{bmatrix}$$

Eg $(n=2)$ $u_1 = -1$ $u_2 = 1$, $A_0 = 1$, $A_1 = 0$ $A_2 = 1$

Then $c_2 = \dfrac{A_1 - A_2}{\pi} = -\dfrac{1}{\pi}$ $c_1 = \dfrac{A_0 - A_1}{\pi} = \dfrac{1}{\pi}$ $c_0 = A_2 = 1$



$$\Phi(u,v) = 1 + \frac{1}{\pi} \operatorname{Arg}(w+1) - \frac{1}{\pi} \operatorname{Arg}(w-1)$$

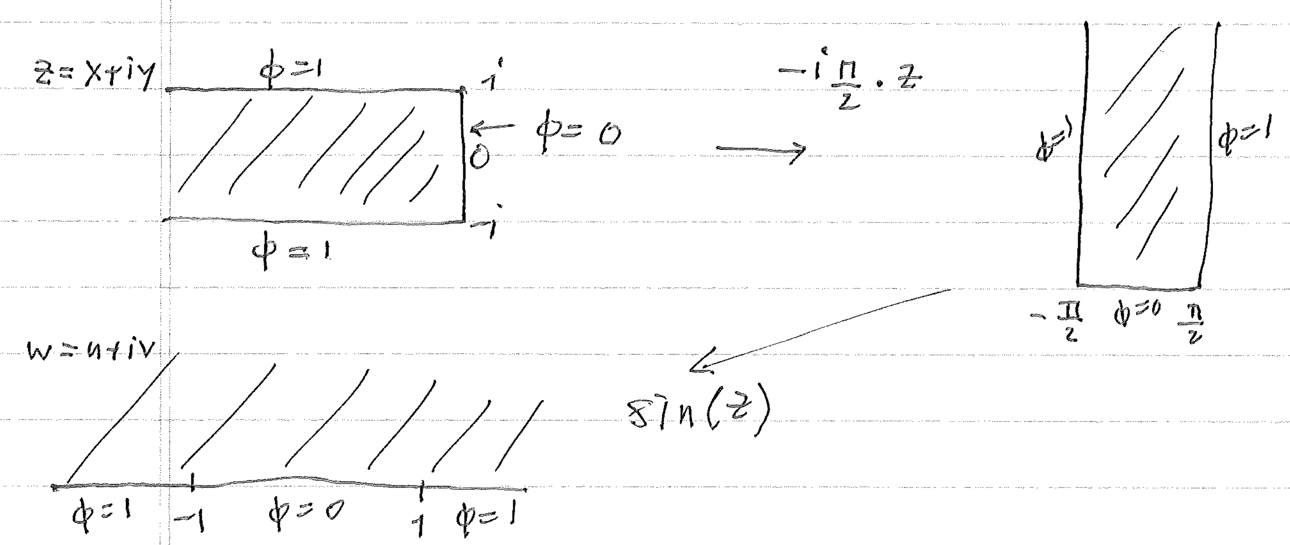
②

Note: $\operatorname{Arg}(w+1) = \operatorname{Arg}(u+iv+1) = \cot^{-1}\left(\frac{u+1}{v}\right)$

$\quad\ldots\; w-1 \;\ldots \qquad = \cot^{-1}\left(\frac{u-1}{v}\right)$

$$\boxed{\Phi(u,v) = 1 + \frac{1}{\pi}\cot^{-1}\left(\frac{u+1}{v}\right) - \frac{1}{\pi}\cot^{-1}\left(\frac{u-1}{v}\right)}$$

Example 1 Solve $\Delta\phi = 0$ with indicated B.C.



$\Phi(u,v)$ as above $\quad f(z) = \sin\left(-\frac{i\pi}{2}\cdot z\right) =$

$\sin\left(\frac{-i\pi}{2}(x+iy)\right) =$

$\sin\left(\frac{\pi y}{2} - i\frac{\pi x}{2}\right)$

$$f(x+iy) = \underbrace{\sin\left(\frac{\pi y}{2}\right)\cosh\left(-\frac{\pi x}{2}\right)}_{u(x,y)} + i\,\underbrace{\cos\left(\frac{\pi y}{2}\right)\sinh\left(-\frac{\pi x}{2}\right)}_{v(x,y)}$$

$$\boxed{\phi(x,y) = \Phi(u(x,y), v(x,y)) \quad \text{is soln}}$$

```
> Phi := proc(u,v) 1 + arccot((u+1)/v)/Pi - arccot((u-1)/v)/Pi end;
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$$\Phi := \mathbf{proc}(u, v)\ \ 1 + \operatorname{arccot}((u+1)\,/\,v)\,/\,\mathrm{Pi} - \operatorname{arccot}((u-1)\,/\,v)\,/\,\mathrm{Pi}\ \mathbf{end\ proc} \tag{1}$$

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> plot3d(Phi(u,v), u=-3..3, v=0..1);
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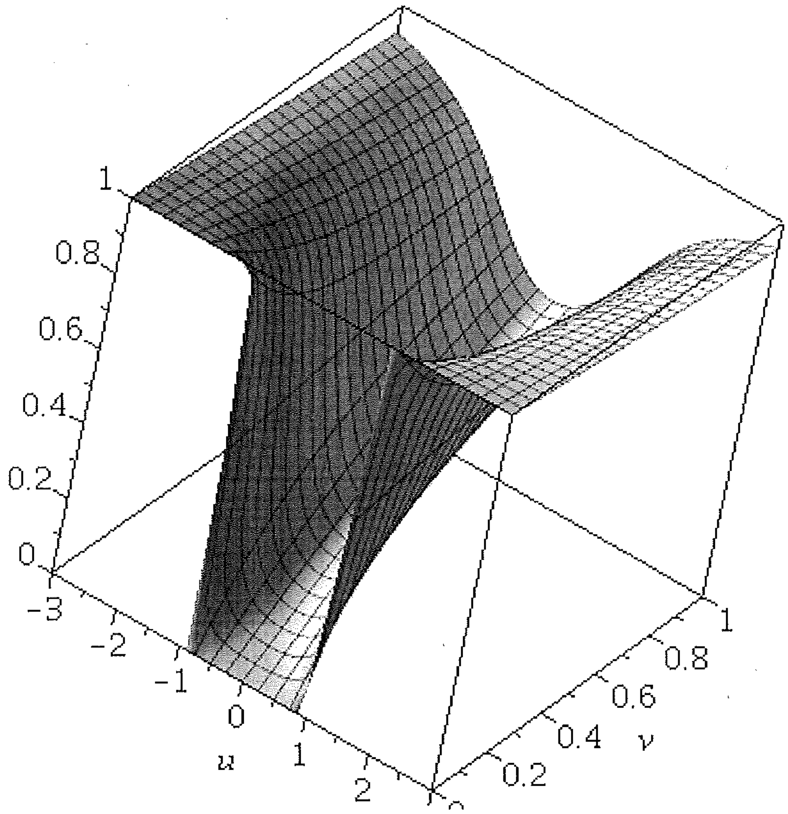
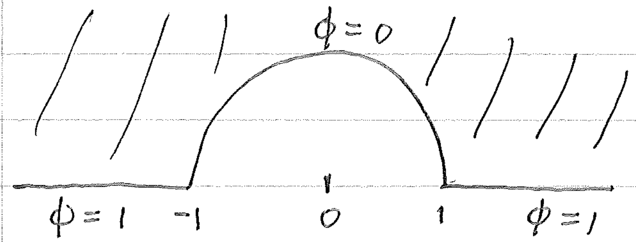


Example 2 Solve $\Delta \phi = 0$ with ...



Joukovski map : $f(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$

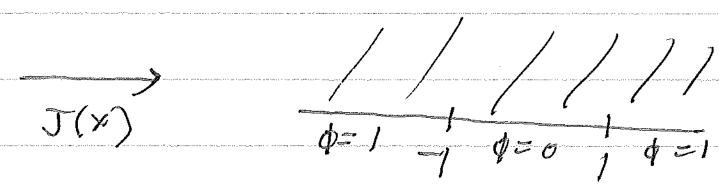
$z = e^{i\theta}$, $\theta \in [0, \pi]$

$J(e^{i\theta}) = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) = \cos(\theta)$, $\theta \in [0, \pi]$

image is a line from 1 to -1

$z = x > 1 \rightarrow J(x)$ maps to $[1, \infty)$

$z = x < -1 \rightarrow J(x)$ maps to $(-\infty, -1]$



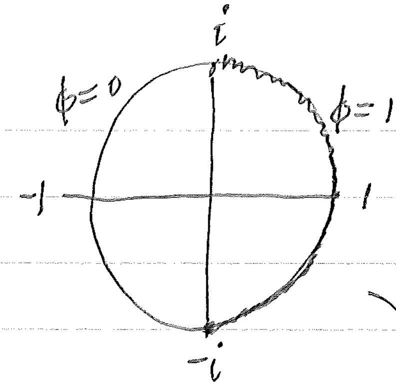
$$J(x+iy) = \frac{1}{2}\left[(x+iy) + \frac{1}{x+iy}\right]$$

$$= \frac{1}{2}\left[x + iy + \frac{x - iy}{x^2+y^2}\right]$$

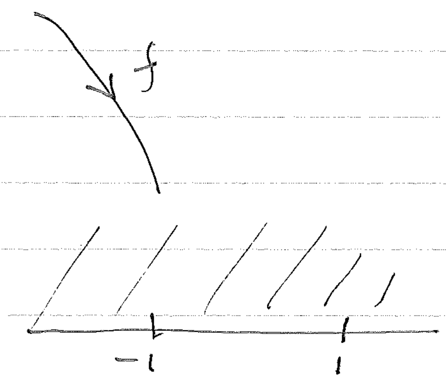
$$= \underbrace{\frac{1}{2}\left(x + \frac{x}{x^2+y^2}\right)}_{u(x,y)} + i\,\underbrace{\frac{1}{2}\left(y - \frac{y}{x^2+y^2}\right)}_{v(x,y)}$$

Soln: $\phi(x,y) = \Phi(u(x,y), v(x,y))$

Example :



Use $f(z) = \dfrac{-iz-i}{z-1}$



also maps to UHP

$$f(x+iy) = \underbrace{\frac{-2y}{x^2+y^2-2x+1}}_{u(x,y)} + i\,\underbrace{\frac{1-x^2-y^2}{x^2+y^2-2x+1}}_{v(x,y)}$$

solution is

$$\phi(x,y) = \Phi\big(u(x,y),\, v(x,y)\big)$$

as above

Poisson Formula

Suppose $f(z)$ is analytic in a region containing the unit disk. Then CIF $\Rightarrow$ for $|z|<1$

$$f(z) = \frac{1}{2\pi i} \oint_{|w|=1} \frac{f(w)}{w-z}\, dw$$

Turn this into a formula that just involves $u = \mathrm{Re}(f)$.

Consider $\dfrac{f(w)\,\bar{z}}{1-w\bar{z}}$. For fixed z with $|z|<1$

this is analytic for $w \in \{|z| \le 1\}$. Thus

$$\frac{1}{2\pi i} \oint_{|w|=1} \frac{f(w)\,\bar{z}}{1-w\bar{z}}\, dw = 0$$

$$f(z) = \frac{1}{2\pi i} \oint_{|w|=1} \underbrace{\left[\frac{1}{w-z} - \frac{\bar{z}}{1-w\bar{z}} \right]}_{\frac{(1-w\bar{z}) - \bar{z}(w-z)}{(w-z)(1-w\bar{z})} \,=\, \frac{1-|z|^2}{(w-z)(1-w\bar{z})}} f(w)\, dw$$

Now parametrize: $w = e^{i\theta}$ $\quad dw = i w\, d\theta$

$$f(z) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{(1-|z|^2)}{(e^{i\theta}-z)(1-e^{i\theta}\bar{z})} f(e^{i\theta})\, i e^{i\theta}\, d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{1-|z|^2}{(e^{i\theta}-z)(\bar{e}^{i\theta}-\bar{z})} f(e^{i\theta})\, d\theta$$

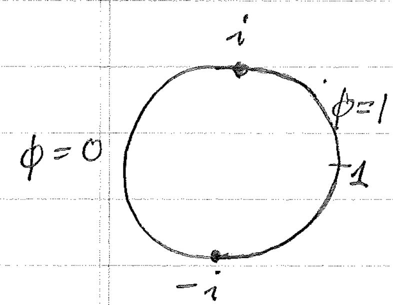
$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{1-|z|^2}{|e^{i\theta}-z|^2} f(e^{i\theta})\, d\theta$$

Now take Re part : $f = u + iv$

$$u(re^{i\varphi}) = \int_0^{2\pi} \frac{1-r^2}{|e^{i\theta}-re^{i\varphi}|^2} u(e^{i\theta})\, d\theta$$

$$u(re^{i\varphi}) = \int_0^{2\pi} \frac{1-r^2}{1+r^2-2r\cos(\theta-\varphi)} u(e^{i\theta})\, d\theta$$

Example $\quad u(e^{i\theta}) = \begin{cases} 0 & \theta \in [\frac{\pi}{2}, \frac{3\pi}{2}] \\ 1 & \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}] \end{cases}$



Method 1 $\quad$ Poisson $\quad u(re^{i\varphi}) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1-r^2}{1+r^2-2r\cos(\theta-\varphi)}\, d\theta$

Method 2 $\quad$ Map to UHP using $f(z) = \frac{-z+i}{-z-i}$

$f : \mathbb{H} \to \mathbb{D}$. $\quad f^{-1}(z) = \frac{-iz-i}{z-1} \quad f^{-1} : \mathbb{D} \to \mathbb{H}$.