Math 301 extra conformal solutions 5

filename: `conformal.extras.solutions.5.tex`

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5. For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $ad - bc \neq 0$, let $f_A(z)$ denote the fractional linear transformation $f_A(z) = (az + b)/(cz + d)$.

(i). Verify that $f_A \circ f_B(z) = f_{AB}(z)$. (Here $f_A \circ f_B(z)$ denotes the composition of maps $f_A(f_B(z))$.)

(ii). Verify that for $\alpha \neq 0$, $f_{\alpha A}(z) = f_A(z)$

(iii). Show that $f^{-1}_{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}(z) = f_{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}(z)$.

Solution:

(i). Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$. Then

$$\begin{aligned} f_A \circ f_B(z) = f_A(f_B(z)) &= f_A\left(\frac{ez+f}{gz+h}\right) \\ &= \left(\frac{a\dfrac{ez+f}{gz+h} + b}{c\dfrac{ez+f}{gz+h} + d}\right) \\ &= \left(\frac{aez + af + bgz + bh}{cez + cfdgz + dh}\right) \\ &= \left(\frac{(ae+bg)z + (af+bh)}{(ce+dg)z + (cf+dh)}\right) \\ &= f_{AB}(z). \end{aligned}$$

(ii)

$$f_{\alpha A}(z) = \left(\frac{\alpha az + \alpha b}{\alpha cz + \alpha d}\right) = \left(\frac{az+b)}{cz+d}\right) = f_A(z)$$

(iii)

$$f_{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}} \circ f_{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} = f_{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} \circ f_{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}} = f_{\alpha I}$$

with $\alpha = ad - bc \neq 0$. We have $f_{\alpha I}(z) = f_I(z) = z$.