Math 301 extra conformal solutions

filename: `extra.conformal.solutions.tex`

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1. Let D be the first quadrant $\{x+iy : x>0, y>0\}$ with the unit (quarter) circle removed. The boundary of D is $C_1 \cup C_2 \cup C_3$ where $C_1 = \{iy : 1 \le y < \infty\}$, $C_2 = \{e^{i\theta} : 0 \le \theta \le \pi/2\}$ and $C_3 = \{x : 1 \le x < \infty\}$. Solve $\Delta\varphi = 0$ in D with $\varphi = 0$ on C_1, $\varphi = 1$ on C_2 and $\varphi = 0$ on C_3. (Hint: use the Joukowski map composed with another map.)

Solution: We can compose the Joukowski map $J(z) = (z+1/z)/2$ with z^2 in either order.
If we apply z^2 first, then D is mapped to the upper half plane with the unit (half) circle removed. This is then mapped to the upper half plane under J. In this case the composite map is $f_1(z) = J(z^2)$. Under this map, C_1 is mapped to $(-\infty,-1]$, C_2 is mapped to $[-1..1]$ and C_3 is mapped to $[1,\infty)$. We must first solve $\Delta\Phi_1(u,v) = 0$ with boundary conditions

$$\Phi_1(u,0) = \begin{cases} 0 & \text{if } u < -1 \\ 1 & \text{if } -1 < u < 1 \\ 0 & \text{if } 1 < u \end{cases}$$

The solution is

$$\Phi_1(u,v) = \frac{1}{\pi}\cot^{-1}\left(\frac{u-1}{v}\right) - \frac{1}{\pi}\cot^{-1}\left(\frac{u+1}{v}\right)$$

Now we write

$$f_1(x+iy) = u_1(x,y) + iv_1(x,y) = \frac{1}{2}\left(x^2-y^2+(x^2-y^2)/(x^2+y^2)^2 + i(xy - xy/(x^2+y^2)^2)\right)$$

which results in the solution

$$\begin{aligned} \varphi_1(x,y) &= \Phi_1(u_1(x,y), v_1(x,y)) \\ &= \frac{1}{\pi}\cot^{-1}(A_1(x,y)) - \frac{1}{\pi}\cot^{-1}(A_2(x,y)) \end{aligned}$$

where

$$A_1(x,y) = \frac{u_1(x,y)-1}{v_1(x,y)} = \frac{(x^3-x^2y+xy^2-y^3-x-y)(x^3+x^2y+xy^2+y^3-x+y)}{2xy(x^2+y^2+1)(x^2+y^2-1)}$$

and

$$A_2(x,y) = \frac{u_1(x,y)+1}{v_1(x,y)} = \frac{(x^3+x^2y+xy^2+y^3+x-y)(x^3-x^2y+xy^2-y^3+x+y)}{2xy(x^2+y^2+1)(x^2+y^2-1)}$$

If we apply $J(z)$ first, then D is mapped to the first quadrant, which is then mapped to the upper half plane under z^2. In this case the composite map is $f_2(z) = J(z)^2$. Under this map, C_1 is mapped to $(-\infty,0]$, C_2 is mapped to $[0..1]$ and C_3 is mapped to $[1,\infty)$. In this case we must first solve $\Delta\Phi_1(u,v) = 0$ with boundary conditions

$$\Phi_1(u,0) = \begin{cases} 0 & \text{if } u < 0 \\ 1 & \text{if } 0 < u < 1 \\ 0 & \text{if } 1 < u \end{cases}$$

The solution is

$$\Phi_2(u,v) = \frac{1}{\pi}\cot^{-1}\left(\frac{u-1}{v}\right) - \frac{1}{\pi}\cot^{-1}\left(\frac{u}{v}\right)$$

Now we write

$$\begin{aligned} f_2(x+iy) &= u_2(x,y) + iv_2(x,y) \\ &= \frac{1}{4}\left(x^2 + 2x^2/(x^2+y^2) + x^2/(x^2+y^2)^2 - y^2 + 2y^2/(x^2+y^2)y^2/(x^2+y^2)^2\right. \\ &\quad \left. + i(2xy - 2xy/(x^2+y^2)^2)\right) \end{aligned}$$

which results in the solution

$$\begin{aligned} \varphi_2(x,y) &= \Phi_2(u_2(x,y), v_2(x,y)) \\ &= \frac{1}{\pi}\cot^{-1}\left(B_1(x,y)\right) - \frac{1}{\pi}\cot^{-1}\left(B_2(x,y)\right) \end{aligned}$$

where

$$B_1(x,y) = \frac{u_2(x,y)-1}{v_2(x,y)} = \frac{(x^3 - x^2y + xy^2 - y^3 - x - y)(x^3 + x^2y + xy^2 + y^3 - x + y)}{2xy(x^2+y^2+1)(x^2+y^2-1)}$$

and

$$B_2(x,y) = \frac{u_2(x,y)}{v_2(x,y)} = \frac{(x^3 + x^2y + xy^2 + y^3 + x - y)(x^3 - x^2y + xy^2 - y^3 + x + y)}{2xy(x^2+y^2+1)(x^2+y^2-1)}$$

Notice that $A_1 = B_1$ and $A_2 = B_2$. This shows that $\varphi_2 = \varphi_1$ so that both methods give the same answer.