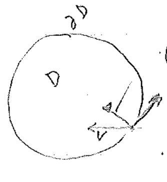
Ideal Fluid Flow in 2-d

- steady (velocity is time independent)
- non viscous (no friction)
- incompressible
- irrotational

(i.e. constant in 3rd dir)

A steady 2-d fluid flow is described by assigning a velocity $V = (V_1, V_2)$ to each point. Assume the flow is incompressible and irrotational.

- incompressible $\leftrightarrow$ net flux into any region is zero



$(-\dot y, \dot x) = n$

$\dot r(t) = (\dot x, \dot y)$

$\int_{\partial D} V\cdot n \, ds = 0$

$$\int_{\partial D} V_1\, dy - V_2\, dx = 0 \qquad = \int_{\partial D} V\cdot n\, ds$$

$n = \frac{(-\dot y, \dot x)}{\sqrt{\dot x^2 + \dot y^2}}$ $\quad ds = \sqrt{\dot x^2+\dot y^2}\, dt$

$$\int_D \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} = 0$$

if $(x(t), y(t))$ is a param. of the boundary

(1) $\frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} = 0$

- irrotational $\leftrightarrow$ circulation around any closed contour is 0

$$\int_{\partial D} V_1\, dx + V_2\, dy = 0 \qquad \int_D \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right) = 0 \qquad (2)\ \frac{\partial V_2}{\partial x} = \frac{\partial V_1}{\partial y}$$

(locally) $\quad$ * could be multivalued

for some function $\Phi(x,y)$

(2) $\Rightarrow$ $V = \nabla \Phi = \left(\frac{\partial \Phi}{\partial x}, \frac{\partial \Phi}{\partial y}\right)$ $\quad$ Then (1) $\Rightarrow$ $\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$, i.e. Φ is (defined up to a const) harmonic. Let Ψ be the harmonic conjugate and define the anal. fcn:

$\Omega(z) = \Phi + i\Psi$ $\quad$ the <u>complex velocity potential</u>

Note $\Omega'(z) = \Phi_x + i\Psi_x = \Phi_x - i\Phi_y$ (by C-R eqns)

$= V_1 - iV_2$

Thus the complex velocity is $V_1 + iV_2 = \overline{\Omega'(z)}$

The speed of the flow is $|\overline{\Omega'(z)}| = |\Omega'(z)|$

Lecture 23

Recall : If $V = (v_1, v_2)$ is the velocity field for an incompressible, irrotational ideal fluid then.

$$V = \nabla \Phi \quad \text{for a harmonic fcn } \Phi(x,y)$$

$\Omega(z) = \Phi + i\Psi$ is the complex velocity potential where Ψ is the harmonic conjugate of Φ. Then

$v_1 + i v_2 = \overline{\Omega'(z)}$ is the complex velocity

T

$A^2 - 4t^2$

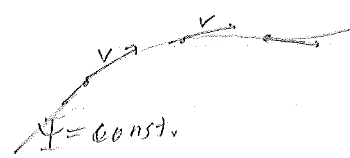
~~stagnation pts $\Leftrightarrow \Omega'(z) = 0$.~~

The function $\Psi(x,y)$ is the <u>stream function</u>. (provided $\Omega'(z) \neq 0$)

We know that
- level curves of Ψ are $\perp$ to level curves of Φ
- $\nabla\Phi$ ($= v$) is also $\perp$ to level curves of Φ

Thus the level curves of the stream fcn Ψ are tangent to v

i.e. they represent the paths of the fluid particles



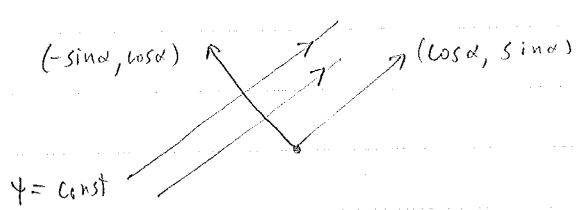
· The points where $\Omega'(z) = 0$ are <u>stagnation points</u>

<u>Example</u> What is the complex velocity potential for steady flow with velocity $v = v_0(\cos\alpha, \sin\alpha)$? ($v_0 > 0$)

$$\overline{\Omega'(z)} = v_0\cos\alpha + i v_0 \sin\alpha = v_0 e^{i\alpha}$$

$$\Omega'(z) = v_0 e^{-i\alpha} \qquad \Omega(z) = v_0 e^{-i\alpha} z \quad (+C).$$

Thus
$$\Phi(x,y) = v_0 \operatorname{Re} e^{-i\alpha} z = v_0 \operatorname{Re}\left((\cos(\alpha) - i\sin(\alpha))(x+iy)\right) = v_0\,(x\cos(\alpha) + y\sin(\alpha))$$
$$\Psi(x,y) = v_0 \operatorname{Im} e^{-i\alpha} z = v_0 \operatorname{Im}(\quad '' \quad) = v_0\,(-x\sin(\alpha) + y\cos(\alpha))$$



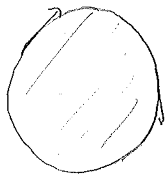
$$\Phi = v_0\, \vec{e}_\alpha \cdot \vec{x} \qquad \vec{e}_\alpha = \begin{bmatrix}\cos\alpha\\ \sin\alpha\end{bmatrix}$$
$$\Psi = v_0\, e_\alpha^\perp \cdot \vec{x} \qquad \vec{x} = \begin{bmatrix}x\\ y\end{bmatrix}$$

$\Psi(\vec{x}) = c$ if the projection of $\vec{x}$ in dn $e_\alpha^\perp$ is const

$$\Psi(x,y) = c \text{ iff } v_0 \begin{bmatrix}-\sin\alpha\\ \cos\alpha\end{bmatrix} \cdot \begin{bmatrix}x\\ y\end{bmatrix} = c$$

Streamlines are $\perp$ to $\begin{bmatrix}-\sin\alpha\\ \cos\alpha\end{bmatrix}$, $\parallel$ to $\begin{bmatrix}\cos(\alpha)\\ \sin(\alpha)\end{bmatrix}$

Fluid Flow around a body $\Omega(z)$ is the complex velocity potential for flow around a body D if the velocity is tangent to boundary



i.e. $\operatorname{Im}\Omega(z) = \Psi(x,y)$ is constant on ∂D.

- any analytic fcn $\Omega = \phi + i\psi$ will be a velocity potential for fluid flow about $\operatorname{Im}\Psi = C$

- If $\Omega(z) \sim wz$ for large z then velocity $= \overline{\Omega'(z)} \sim \overline{w}$ for large z.

Example : Show The Joukowski map $\Omega(z) = V_0\left(z + \frac{a^2}{z}\right)$ $(V_0 > 0,\ a > 0)$

describes fluid flow about a cylinder of radius a

$$\Omega(ae^{i\theta}) = V_0(ae^{i\theta} + ae^{-i\theta}) = 2aV_0\cos\theta$$

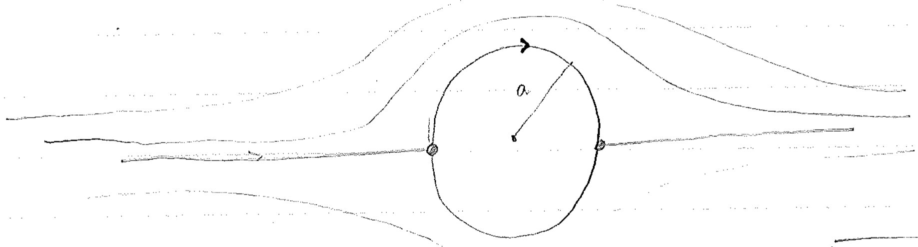
$$\operatorname{Im}(\Omega(ae^{i\theta})) = 0$$

$y\left(1 - \frac{a^2}{x^2+y^2}\right) = C$

$$\operatorname{Im}\Omega(z) = V_0\left(y - \frac{a^2 y}{x^2+y^2}\right)$$ streamlines : Let $z = re^{i\theta}$

$$\operatorname{Im}(\Omega(re^{i\theta})) = V_0 \cdot \sin(\theta)\left(r - \frac{a^2}{r}\right)$$ $(z = re^{i\theta} \to y = r\sin\theta$

$$\sin(\theta)\left(r - \frac{a^2}{r}\right) = C$$ e.g. $r = a$, $\theta = 0$



$$r^2 - \frac{cr}{\sin\theta} - a^2 = 0 \qquad r = \frac{c}{2\sin\theta} + \sqrt{\frac{c^2}{4\sin^2\theta} + a^2}$$ (choose + since $r > 0$) $c > 0$ since $r > a$

complex velocity $\Omega'(z) = V_0\left(1 - \frac{a^2}{z^2}\right)$ $\qquad \overline{\Omega'(z)} = V_0\left(1 - \frac{a^2}{\overline{z}^2}\right)$

Note : $\boxed{\text{tends to } V_0 \text{ as } z \to \infty}$

$$\overline{\Omega'(ia)} = V_0\left(1 - \frac{a^2}{(-ia)^2}\right) = 2V_0$$

stagnation pts : $\overline{z} = \pm a \to z = \pm a$

Flow around cylinder with superimposed vortex

Example: Notice that both $\Omega_1(z) = V_0\left(z + \frac{a^2}{z}\right)$ and $\Omega_2(z) = i\gamma \log(z)$ represent flows around $|z| = a$, i.e. $\mathrm{Im}(\Omega_1(z)) = \text{const}$ and $\mathrm{Im}(\Omega_2(z)) = \text{const}$ when $|z| = a$. Thus

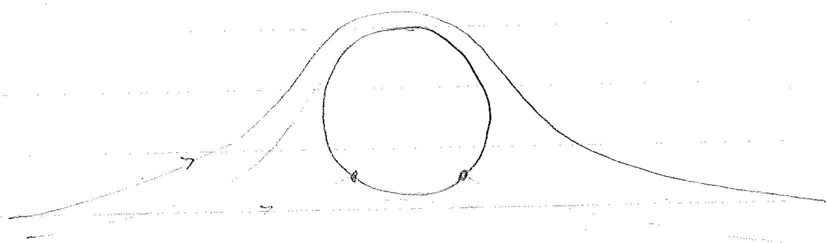
$\Omega(z) = \Omega_1(z) + \Omega_2(z) = V_0\left(z + \frac{a^2}{z}\right) + i\gamma \log(z)$ also represents a flow around a cylinder.

Stagnation points: $\Omega'(z) = V_0\left(1 - \frac{a^2}{z^2}\right) + \frac{i\gamma}{z} = 0$ when

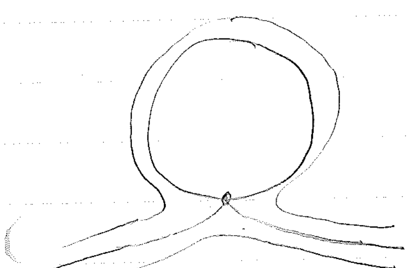
$$z^2 + \frac{i\gamma z}{V_0} - a^2 = 0 \qquad z_\pm = -\frac{i\gamma}{2V_0} \pm \sqrt{-\frac{\gamma^2}{4V_0^2} + a^2}$$

Case 1 $|\gamma| < 2V_0 a$ then $|z_\pm|^2 = \frac{\gamma^2}{4V_0^2} + \left(-\frac{\gamma^2}{4V_0^2} + a^2\right) = a^2$

and there are 2 stagnation pts on unit circle



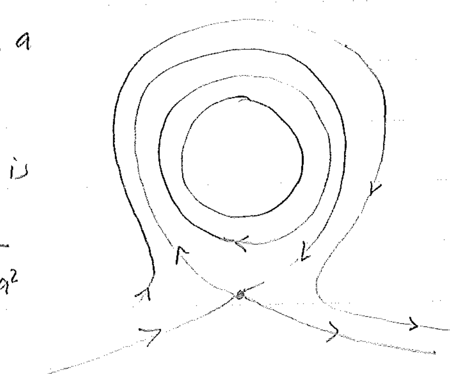
Case 2 $|\gamma| = 2V_0 a$ stagnation points come together



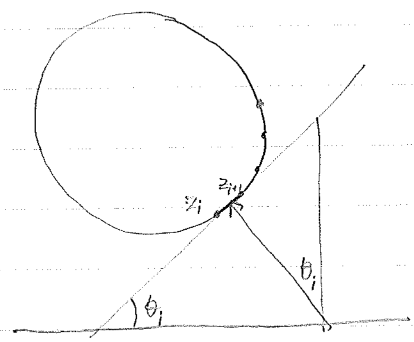
Case 3 $\gamma > 2V_0 a$

one stag. pt on Im. axis (other one is inside).

$$z = -\frac{i\gamma}{2V_0} - i\sqrt{\frac{\gamma^2}{4V_0^2} - a^2}$$



Force exerted on obstacle by fluid pressure



If P_i is the pressure at z_i

$\Delta s_i = |z_{i+1} - z_i|$

$\Delta z_i = z_{i+1} - z_i = e^{i\theta_i}\Delta s_i$

Then the force exerted on the ith segment has magnitude $P_i \Delta s_i$ and direction normal to curve, i.e. $ie^{i\theta_i}$

$\Delta F_i = ie^{i\theta_i} P_i \Delta s_i$ so the total force is $F \cong \sum_i \Delta F_i$

Bernoulli's equation: $P + \frac{1}{2}\rho |V|^2 = \alpha$ ← const along stream lines

↑ pressure ↑ density ↑ velocity.

$$F \cong \sum_i ie^{i\theta_i}\left(\alpha - \tfrac{1}{2}\rho|V_i|^2\right)\Delta s_i = \underbrace{\sum_i i\alpha \Delta z_i}_{(1)} + \underbrace{\sum_i -ie^{i\theta_i}\frac{\rho}{2}|V_i|^2 \Delta s_i}_{(2)}$$

$$(1) \to i\alpha \int_{\partial D} dz = 0$$

Now the complex velocity $\overline{\Omega'(z_i)} = e^{i\theta_i}|V_i|$ (direction tangent).

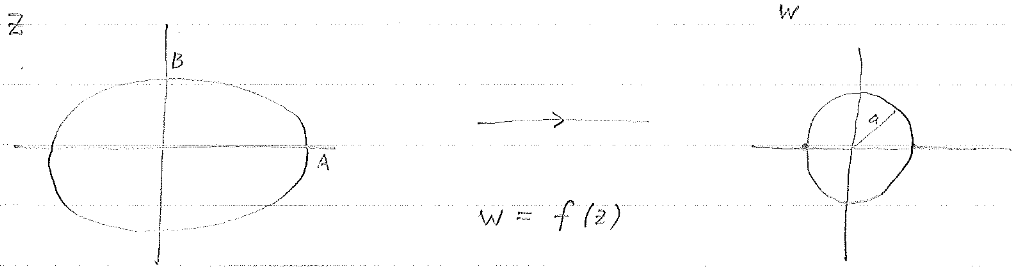
$$\overline{F} \cong \frac{i\rho}{2}\sum_i e^{-i\theta_i}|V_i|^2 \Delta s_i \cong \frac{i\rho}{2}\sum_i \underbrace{e^{-2i\theta_i}|V_i|^2}_{\Omega'(z_i)^2} \underbrace{e^{i\theta_i}\Delta s_i}_{\Delta z_i}$$

$$\Rightarrow \boxed{\overline{F} = \frac{i\rho}{2}\int_{\partial D} \Omega'(z)^2\, dz}$$

Example: When $\Omega(z) = v_0\left(z + \frac{a^2}{z}\right)$ (flow around circle)

$$\overline{F} = \frac{i\rho}{2}\oint_{|z|=a} v_0^2\left(1 - \frac{a^2}{z^2}\right)^2 dz = 0$$

Example Fluid flow around an ellipse; find complex velocity potential



Need to find f(z) mapping ext. of ellipse to ext of circle then $\Omega(f(z))$ is our complex vel. pot.

$$\Omega(w) = V_0\left(w + \frac{a^2}{w}\right)$$

Use Joukowski map again to find $f(z)$. Recall that

$J_\tau(w) = \left(w + \frac{\tau^2}{w}\right)$ maps $|w| = a$ onto an ellipse: $(a \geq \tau)$

$$J(ae^{i\theta}) = \left(ae^{i\theta} + \frac{\tau^2}{a}e^{-i\theta}\right) = \left(a + \frac{\tau^2}{a}\right)\cos\theta + i\left(a - \frac{\tau^2}{a}\right)\sin\theta$$

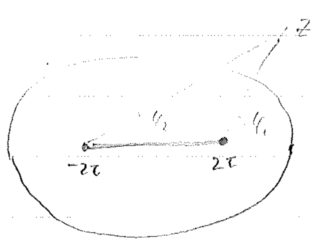
$$A = a + \frac{\tau^2}{a} \qquad B = a - \frac{\tau^2}{a} \qquad a = \frac{A+B}{2} \qquad \frac{A-B}{2} = \frac{\tau^2}{a} \qquad \tau^2 = \frac{(A-B)(A+B)}{4}$$

$$\tau^2 = \frac{A^2 - B^2}{4}$$

$w = f(z)$ is the inverse of J : $z = w + \frac{\tau^2}{w}$ $\qquad w^2 - wz + \tau^2 = 0$

$w = \frac{z}{2} + \frac{1}{2}\left(z^2 - (2\tau)^2\right)^{\frac{1}{2}}$ — branch pts at $\pm 2\tau$. Choose branch cut from -2τ to 2τ, and a branch that is + for $z \in [2\tau, \infty)$ and − for $z \in (-\infty, -2\tau]$.

(then $A \mapsto \frac{A}{2} + \frac{1}{2}B = a$ etc)



$$\Omega(f(z)) = V_0\left(\frac{z}{2} + \frac{1}{2}\left(z^2 - (2\tau)^2\right)^{\frac{1}{2}} + \frac{a^2}{z/2 + \frac{1}{2}\left(z^2 - (2\tau)^2\right)^{\frac{1}{2}}}\right)$$

$$\left(z^2 - (2\tau)^2\right)^{\frac{1}{2}} = \left|z^2 - (2\tau)^2\right|^{\frac{1}{2}} e^{i(\varphi_1 + \varphi_2)/2} \qquad \varphi_i \in [-\pi, \pi]$$

what is the speed of flow at points ~~A~~, iB? $\qquad \Omega'(w) = V_0\left(1 - \frac{a^2}{w^2}\right)$

$$J'(w) = 1 - \frac{\tau^2}{w^2}$$

$$\frac{d}{dz}\left[\Omega(f(z))\right] = \Omega'(f(z))\, f'(z) = \Omega'(f(z))\frac{1}{J'(f(z))}$$

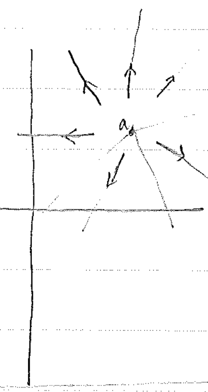
$z = iB \to f(z) = ia$ $\qquad \left|\frac{\Omega'(ia)}{J'(ia)}\right| = \frac{V_0\,|1+1|}{1 + \tau^2/a^2} = \frac{2V_0}{1 + \frac{A^2 - B^2}{(A+B)^2}} = \frac{2V_0\,(A+B)}{A+B+A-B} = V_0\left(1 + \frac{B}{A}\right)$

Sources, sinks and vortex flow

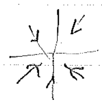
The multivalued fcn $\Omega(z) = k\log(z-a)$ still describes an ideal fluid flow, ~~away from~~ for $z \neq a$, since $\Omega'(z)$ and thus the velocity field $\overline{\Omega'(z)}$ is single valued.

case $\boxed{k>0}$. Then $\Omega'(z) = \dfrac{k}{z-a} = \dfrac{k(\bar{z}-\bar{a})}{|z-a|^2}$ so the complex velocity is $\overline{\Omega'(z)} = k\dfrac{z-a}{|z-a|^2}$

represents a source at $z=a$



case $\boxed{k<0}$ represents a sink



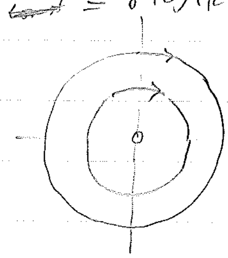
The streamlines are $\operatorname{Im}\Omega(z) = \text{const}$, ~~ie~~ $= \operatorname{Im}(k\log|z-a| + ik\arg(z-a)) = \text{const}$, $k\arg(z-a) = \text{const}$

case $\boxed{k = i\gamma,\ \gamma>0}$ vortex flow (clockwise)

The streamlines are $\operatorname{Im}\left(i\gamma\log(|z-a|) - \gamma\arg(z-a)\right) = \cancel{\text{const}} = \gamma\log(|z-a|) = \text{const}$.

complex velocity : $\Omega'(z) = \dfrac{i\gamma}{z-a} = \dfrac{i\gamma(\bar{z}-\bar{a})}{|z-a|^2}$, so

$\overline{\Omega'(z)} = -i\gamma\dfrac{z-a}{|z-a|^2}$.



case $\boxed{k = i\gamma,\ \gamma<0}$ vortex flow counterclockwise