4. Let f have an isolated singularity at z_0 (f analytic in a punctured neighborhood of z_0). Show that the residue of the derivative f' at z_0 is equal to zero.

5. Is there a function f having a simple pole at z_0 with $\text{Res}(f; z_0) = 0$? How about a function with a pole of order 2 at z_0 and $\text{Res}(f; z_0) = 0$?

6. Suppose that f is analytic and has a zero of order m at the point z_0. Show that the function $g(z) = f'(z)/f(z)$ has a simple pole at z_0 with $\text{Res}(g; z_0) = m$.

7. Evaluate

$$\oint_{|z|=1} e^{1/z} \sin(1/z)\, dz.$$

6.2 Trigonometric Integrals over $[0, 2\pi]$

Our goal in this section is to apply the residue theorem to evaluate real integrals of the form

$$\int_0^{2\pi} U(\cos\theta, \sin\theta)\, d\theta, \tag{1}$$

where $U(\cos\theta, \sin\theta)$ is a rational function (with real coefficients) of $\cos\theta$ and $\sin\theta$ and is finite over $[0, 2\pi]$. An example of such an integral is

$$\int_0^{2\pi} \frac{\sin^2\theta}{5 + 4\cos\theta}\, d\theta\ .$$

We shall show that (1) can be identified as the parametrized form of a contour integral, $\int_C F(z)\, dz$, of some complex function F around the positively oriented unit circle $C : |z| = 1$. To establish this identification we parametrize C by

$$z = e^{i\theta} \qquad (0 \le \theta \le 2\pi).$$

For such z we have

$$\frac{1}{z} = \frac{1}{e^{i\theta}} = e^{-i\theta},$$

and since

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i},$$

we have the identities[†]

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right), \qquad \sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right). \tag{2}$$

Furthermore, when integrating along C,

$$dz = ie^{i\theta}\, d\theta = iz\, d\theta,$$

[†] Of course we could use $\bar{z}$ instead of $1/z$, but this would forfeit analyticity.