If n is a nonzero integer, the reader should have no trouble showing that $z = n\pi$ is a *simple* zero for the given function.

Near the point $z = 0$ we can write

$$\begin{aligned}\frac{\tan z}{z} &= \frac{\sin z}{z \cos z} \\ &= \frac{1}{z\cos z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right) \\ &= \frac{1}{\cos z}\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\right),\end{aligned}$$

and we see that $(\tan z)/z \to 1$ as $z \to 0$. Hence the origin is a *removable singularity*.

Finally, since $\cos z$ has simple zeros at $z = (n + \frac{1}{2})\pi$ for $n = 0, \pm1, \pm2, \ldots$, it is easy to see that $f(z)$ has simple poles at these points. ■

Theorem 18 summarizes the various equivalent characterizations of the three types of isolated singularities. For economy of notation we employ the logician's symbol "$\Leftrightarrow$" to denote logical equivalence; it can be translated "if and only if."

Theorem 18. If f has an isolated singularity at z_0, then the following equivalences hold:

(i) z_0 is a removable singularity $\Leftrightarrow$ $|f|$ is bounded near z_0 $\Leftrightarrow$ $f(z)$ has a limit as $z \to z_0$ $\Leftrightarrow$ f can be redefined at z_0 so that f is analytic at z_0.

(ii) z_0 is a pole $\Leftrightarrow$ $|f(z)| \to \infty$ as $z \to z_0$ $\Leftrightarrow$ f can be written $f(z) = g(z)/(z - z_0)^m$ for some integer $m > 0$ and some function g analytic at z_0 with $g(z_0) \neq 0$.

(iii) z_0 is an essential singularity $\Leftrightarrow$ $|f(z)|$ neither is bounded near z_0 nor goes to infinity as $z \to z_0$ $\Leftrightarrow$ $f(z)$ assumes every complex number, with possibly one exception, as a value in every neighborhood of z_0.

In closing, we make a few some general observations. Earlier we saw that the seemingly innocent-looking property of analyticity for a function f at a point z_0 places enormous restrictions on f; in particular, it must be infinitely differentiable, and expressible by its Taylor series in a neighborhood of z_0. Now we find that if f is merely presumed to be defined, and analytic, in a *punctured* neighborhood of z_0 (like $0 < |z - z_0| < r$), then it is still strongly restricted. One can characterize its behavior near z_0 by asking how many powers of $(z - z_0)$ would it take to "civilize" $f(z)$, in the sense that $(z - z_0)^m f(z)$ would have a finite, *nonzero* limiting value as $z \to z_0$. If the answer (m) is a positive integer, then f has a pole of order m at z_0 and it can be written as $g(z)/(z - z_0)^m$ with g analytic and nonzero at z_0. If m is a negative integer, then f can be written as $g(z)(z - z_0)^{|m|}$ with g, again, analytic and nonzero at z_0; the latter form exhibits a zero of order $|m|$ at z_0. If m is zero, then f has a removable singularity at z_0.

The only other possibility is that no such m exists, that is, no power of $(z - z_0)$ can endow $(z - z_0)^m f(z)$ with a nonzero limit at z_0. Then unless f is identically zero (and not worth "civilizing"), it has an essential singularity at z_0, taking *all* complex numbers as values in any neighborhood of z_0 (with, possibly, one exception).

EXERCISES 5.6

1. Find and classify the isolated singularities of each of the following functions.

(a) $\dfrac{z^3+1}{z^2(z+1)}$ **(b)** $z^3 e^{1/z}$ **(c)** $\dfrac{\cos z}{z^2+1} + 4z$ **(d)** $\dfrac{1}{e^z - 1}$

(e) $\tan z$ **(f)** $\cos\left(1 - \dfrac{1}{z}\right)$ **(g)** $\dfrac{\sin(3z)}{z^2} - \dfrac{3}{z}$ **(h)** $\cot\left(\dfrac{1}{z}\right)$

2. What is the order of the pole of

$$f(z) = \frac{1}{(2\cos z - 2 + z^2)^2}$$

at $z = 0$? [HINT: Work with $1/f(z)$.]

3. For each of the following, construct a function f, analytic in the plane except for isolated singularities, that satisfies the given conditions.

(a) f has a zero of order 2 at $z = i$ and a pole of order 5 at $z = 2 - 3i$.

(b) f has a simple zero at $z = 0$ and an essential singularity at $z = 1$.

(c) f has a removable singularity at $z = 0$, a pole of order 6 at $z = 1$, and an essential singularity at $z = i$.

(d) f has a pole of order 2 at $z = 1 + i$ and essential singularities at $z = 0$ and $z = 1$.

4. Give a proof of Lemma 8.

5. For each of the following, determine whether the statement made is always true or sometimes false.

(a) If f and g have a pole at z_0, then $f + g$ has a pole at z_0.

(b) If f has an essential singularity at z_0 and g has a pole at z_0, then $f + g$ has an essential singularity at z_0.

(c) If $f(z)$ has a pole of order m at $z = 0$, then $f(z^2)$ has a pole of order $2m$ at $z = 0$.

(d) If f has a pole at z_0 and g has an essential singularity at z_0, then the product $f \cdot g$ has a pole at z_0.

(e) If f has a zero of order m at z_0 and g has a pole of order n, $n \leq m$, at z_0, then the product $f \cdot g$ has a removable singularity at z_0.

6. Prove that if $f(z)$ has a pole of order m at z_0, then $f'(z)$ has a pole of order $m + 1$ at z_0.

7. If $f(z)$ is analytic in $D : 0 < |z| \leq 1$, and $z^\ell \cdot f(z)$ is unbounded in D for every integer ℓ, then what kind of singularity does $f(z)$ have at $z = 0$?

8. Verify Picard's theorem for the function $\cos(1/z)$ at $z_0 = 0$.

9. Does there exist a function $f(z)$ having an essential singularity at z_0 that is bounded along some line segment emanating from z_0?

10. If the function $f(z)$ is analytic in a domain D and has zeros at the distinct points $z_1, z_2, \ldots, z_n$ of respective orders $m_1, m_2, \ldots, m_n$, then prove that there exists a function $g(z)$ analytic in D such that

$$f(z) = (z - z_1)^{m_1} (z - z_2)^{m_2} \cdots (z - z_n)^{m_n} g(z).$$

11. If f has a pole at z_0, show that Re f and Im f take on arbitrarily large positive as well as negative values in any punctured neighborhood of z_0.

12. Prove that if $f(z)$ has a pole of order m at z_0, then $g(z) := f'(z)/f(z)$ has a simple pole at z_0. What is the coefficient of $(z - z_0)^{-1}$ in the Laurent expansion for $g(z)$?

13. Let $f(z)$ have an isolated singularity at z_0 and suppose that $f(z)$ is bounded in some punctured neighborhood of z_0. Prove directly from the integral formula for the Laurent coefficients that $a_{-j} = 0$ for all $j = 1, 2, \ldots$; that is, $f(z)$ must have a removable singularity at z_0.

14. Without appealing to Picard's theorem, prove the theorem of *Casorati* and *Weierstrass*:† If $f(z)$ has an essential singularity at z_0, then in any punctured neighborhood of z_0 the function $f(z)$ comes arbitrarily close to any specified complex number. [HINT: Let the specified number be c and assume to the contrary that $|f(z) - c| \geq \delta > 0$ in every small punctured neighborhood of z_0. Then, using Prob. 13, show that $f(z) - c$ [and hence $f(z)$ itself] must have either a pole or a removable singularity at z_0.]

15. Prove that if $f(z)$ has an essential singularity at z_0, then so does the function $e^{f(z)}$. [HINT: Argue that $e^{f(z)}$ is neither bounded nor tends (in modulus) to infinity as $z \to z_0$.]

16. Sketch the graphs for $s = 1, \frac{1}{2}, 2, \frac{1}{3}, 3, \ldots$ of the level curves $\left|e^{1/z}\right| = s$, and observe that they all converge at the essential singularity $z = 0$ of $e^{1/z}$. [HINT: The level curves are all circles.]

17. By completing each of the following steps, prove Schwarz's lemma.

†Felice Casorati (1835–1890), Karl Theodor Wilhelm Weierstrass (1853–1897). In his lectures developing the subject of analysis, Weierstrass established standards of rigor for the future of mathematics.

(d) If f has a pole at z_0 and g has an essential singularity at z_0, then the product $f \cdot g$ has a pole at z_0.

(e) If f has a zero of order m at z_0 and g has a pole of order n, $n \le m$, at z_0, then the product $f \cdot g$ has a removable singularity at z_0.

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15. Prove that if $f(z)$ has an essential singularity at z_0, then so does the function $e^{f(z)}$. [HINT: Argue that $e^{f(z)}$ is neither bounded nor tends (in modulus) to infinity as $z \to z_0$.]

16. Sketch the graphs for $s = 1, \frac{1}{2}, 2, \frac{1}{3}, 3, \ldots$ of the level curves $|e^{1/z}| = s$, and observe that they all converge at the essential singularity $z = 0$ of $e^{1/z}$. [HINT: The level curves are all circles.]

17. By completing each of the following steps, prove Schwarz's lemma.

[†]Felice Casorati (1835–1890), Karl Theodor Wilhelm Weierstrass (1853–1897). In his lectures developing the subject of analysis, Weierstrass established standards of rigor for the future of mathematics.

(*Schwarz's Lemma*) If f is analytic in the unit disk $U: |z| < 1$ and satisfies the conditions

$$f(0) = 0 \text{ and } |f(z)| \le 1 \text{ for all } z \text{ in } U,$$

then $|f(z)| \le |z|$ for all z in U.

(a) Define $F(z) := f(z)/z$, for $z \ne 0$, and $F(0) = f'(0)$. Show that F is analytic in U.

(b) Let $\zeta (\ne 0)$ be any fixed point in U, and r be any real number that satisfies $|\zeta| < r < 1$. Show by means of the maximum-modulus principle that if C_r denotes the circle $|z| = r$, then

$$|F(\zeta)| \le \max_{z \text{ on } C_r} \frac{|f(z)|}{r} \le \frac{1}{r}.$$

(c) Letting $r \to 1^-$ in part (b), deduce that $|f(\zeta)| \le |\zeta|$ for all ζ in U.

18. Let f be a function satisfying the conditions of Schwarz's lemma (Prob. 17). Prove that if $|f(z_0)| = |z_0|$ for some nonzero z_0 in U, then f must be a function of the form $f(z) = e^{i\theta}z$ for some real θ. Show also that f must be of this form if $|f'(0)| = 1$.

19. Define the function $h(z)$ by

$$h(z) = \frac{1}{\sin z} - \frac{1}{z} + \frac{2z}{z^2 - \pi^2}.$$

(a) Show that $h(z)$ is analytic in the disk $|z| < 2\pi$, except for removable singularities at $z = 0, \pm\pi$.

(b) Find the first four terms of the Taylor series about $z = 0$ for $h(z)$. What is the radius of convergence of this series?

(c) Use the result of part (b) to obtain the first few coefficients (with positive *and* negative indices) in the Laurent series expansion for $\csc z = 1/\sin z$, valid in the annulus $\pi < |z| < 2\pi$.

5.7 The Point at Infinity

From our discussion of singularities in Sec. 5.6 we know that if a mapping is given by an analytic function possessing a pole, it carries points near that pole to indefinitely distant points. It must have occurred to the reader that one might take the value of f *at* the pole to be ∞. Before taking this plunge, however, we should be aware of all the ramifications. Let us look in detail at the behavior of $1/z$ near $z = 0$.

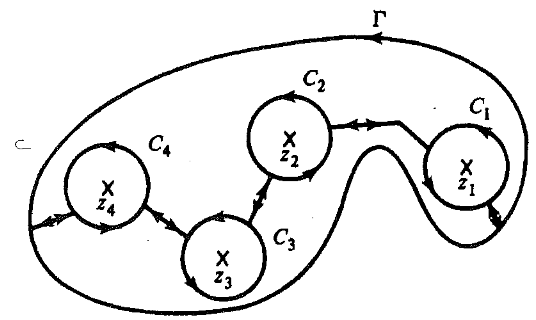

Figure 6.3 Equivalent contours for integration.

Hence we have established the following important result.

Theorem 2 (*Cauchy's Residue Theorem*). If Γ is a simple closed positively oriented contour and f is analytic inside and on Γ except at the points $z_1, z_2, \ldots, z_n$ inside Γ, then

$$\int_{\Gamma} f(z)\,dz = 2\pi i \sum_{j=1}^{n} \operatorname{Res}(z_j). \tag{5}$$

Example 5

Evaluate

$$\oint_{|z|=2} \frac{1-2z}{z(z-1)(z-3)}\,dz.$$

Solution. The integrand $f(z) = (1-2z)/[z(z-1)(z-3)]$ has simple poles at $z = 0$, $z = 1$, and $z = 3$. However, only the first two of these points lie inside $\Gamma : |z| = 2$. Thus by the residue theorem

$$\oint_{|z|=2} f(z)\,dz = 2\pi i[\operatorname{Res}(0) + \operatorname{Res}(1)],$$

and since

$$\operatorname{Res}(0) = \lim_{z\to 0} z\,f(z) = \lim_{z\to 0} \frac{1-2z}{(z-1)(z-3)} = \frac{1}{3},$$
$$\operatorname{Res}(1) = \lim_{z\to 1} (z-1) f(z) = \lim_{z\to 1} \frac{1-2z}{z(z-3)} = \frac{1}{2},$$

we obtain

$$\oint_{|z|=2} f(z)\,dz = 2\pi i \left(\frac{1}{3} + \frac{1}{2}\right) = \frac{5\pi i}{3}. \quad \blacksquare$$

Example 6

Compute

$$\oint_{|z|=5} \left[z e^{3/z} + \frac{\cos z}{z^2(z-\pi)^3} \right] dz.$$

Solution. The given integral can obviously be expressed as the sum

$$\oint_{|z|=5} z e^{3/z}\,dz + \oint_{|z|=5} \frac{\cos z}{z^2(z-\pi)}\,dz,$$

which, by the residue theorem, equals

$$2\pi i \left[\operatorname{Res}\left(z e^{3/z}; 0\right) + \operatorname{Res}\left(\frac{\cos z}{z^2(z-\pi)^3}; 0\right) + \operatorname{Res}\left(\frac{\cos z}{z^2(z-\pi)^3}; \pi\right) \right].$$

These residues were computed in Examples 1 and 4; the desired answer is therefore

$$2\pi i \left[\frac{9}{2} - \frac{3}{\pi^4} - \frac{(6-\pi^2)}{2\pi^4} \right]. \quad \blacksquare$$

EXERCISES 6.1

1. Determine all the isolated singularities of each of the following functions and compute the residue at each singularity.

(a) $\dfrac{e^{3z}}{z-2}$ **(b)** $\dfrac{z+1}{z^2-3z+2}$ **(c)** $\dfrac{\cos z}{z^2}$ **(d)** $\left(\dfrac{z-1}{z+1}\right)^3$

(e) $\dfrac{e^z}{z(z+1)^3}$ **(f)** $\sin\left(\dfrac{1}{3z}\right)$ **(g)** $\tan z$ **(h)** $\dfrac{z-1}{\sin z}$

(i) $z^2/(1-\sqrt{z})$, where $\sqrt{z}$ denotes the principal branch.

2. Explain why Cauchy's integral formula can be regarded as a special case of the residue theorem.

3. Evaluate each of the following integrals by means of the Cauchy residue theorem.

(a) $\displaystyle\oint_{|z|=5} \frac{\sin z}{z^2-4}\,dz$ **(b)** $\displaystyle\oint_{|z|=3} \frac{e^z}{z(z-2)^3}\,dz$

(c) $\displaystyle\oint_{|z|=2\pi} \tan z\,dz$ **(d)** $\displaystyle\oint_{|z|=3} \frac{e^{iz}}{z^2(z-2)(z+5i)}\,dz$

(e) $\displaystyle\oint_{|z|=1} \frac{1}{z^2 \sin z}\,dz$ **(f)** $\displaystyle\oint_{|z|=3} \frac{3z+2}{z^4+1}\,dz$

(g) $\displaystyle\oint_{|z|=8} \frac{1}{z^2+z+1}\,dz$

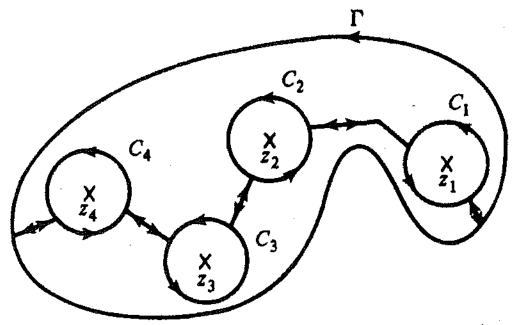

Figure 6.3 Equivalent contours for integration.

Hence we have established the following important result.

> **Theorem 2** (***Cauchy's Residue Theorem***). If Γ is a simple closed positively oriented contour and f is analytic inside and on Γ except at the points $z_1, z_2, \ldots, z_n$ inside Γ, then
>
> $$\int_\Gamma f(z)\,dz = 2\pi i \sum_{j=1}^{n} \operatorname{Res}(z_j). \tag{5}$$

Example 5

Evaluate

$$\oint_{|z|=2} \frac{1-2z}{z(z-1)(z-3)}\,dz.$$

Solution. The integrand $f(z) = (1-2z)/[z(z-1)(z-3)]$ has simple poles at $z = 0$, $z = 1$, and $z = 3$. However, only the first two of these points lie inside $\Gamma : |z| = 2$. Thus by the residue theorem

$$\oint_{|z|=2} f(z)\,dz = 2\pi i[\operatorname{Res}(0) + \operatorname{Res}(1)],$$

and since

$$\operatorname{Res}(0) = \lim_{z\to 0} z\,f(z) = \lim_{z\to 0} \frac{1-2z}{(z-1)(z-3)} = \frac{1}{3},$$

$$\operatorname{Res}(1) = \lim_{z\to 1} (z-1)\,f(z) = \lim_{z\to 1} \frac{1-2z}{z(z-3)} = \frac{1}{2},$$

we obtain

$$\oint_{|z|=2} f(z)\,dz = 2\pi i\left(\frac{1}{3} + \frac{1}{2}\right) = \frac{5\pi i}{3}. \quad \blacksquare$$