Math 301 Hmk 1

5.6 p. 285   1adeg, 5abcd, 12, 13, 14, 15

1a $\frac{z^3+1}{z^2(z+1)}$ $z_0 = 0$ is a pole of order 2. $z_0 = -1$ is removable since $z^3+1 = (z+1)(z^2-z+1)$ so the $z+1$ term cancels.

1d $\frac{1}{e^z-1}$ $z_0 = 2\pi i k$, $k \in \mathbb{Z}$ are simple poles

1e $\tan z = \frac{\sin z}{\cos z}$ $z_0 = \frac{\pi}{2} + \pi k$, $k \in \mathbb{Z}$ are simple poles

1g $\frac{\sin(3z)}{z^2} - \frac{3}{z} = \frac{3z - (3z)^3/3! + \cdots}{z^2} - \frac{3}{z} = -\frac{27}{3!} z + \cdots$

so $z_0 = 0$ is a removable singularity.

5a False, eg $f = \frac{1}{z-z_0}$, $g = \frac{-1}{z-z_0}$, $f+g = 0$.

5b True, adding g can change only finitely many negative coeficients in the Laurent series

5c True, $f(z) = \frac{a_{-m}}{z^m} + \frac{a_{-m+1}}{z^{m-1}} + \cdots$ so $f(z^2) = \frac{a_{-m}}{z^{2m}} + \frac{a_{-m+1}}{z^{2m-2}} +$
(with $a_{-m} \neq 0$)

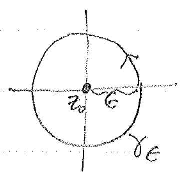
5d False, $f(z) = \frac{1}{(z-z_0)^m} h(z)$ when $h(z)$ is analytic and $h(z_0) \neq 0$.
If $f \cdot g$ has a pole then $f(z)g(z) = \frac{1}{(z-z_0)^n} k(z)$ where k is analytic near z_0
so $g(z) = \frac{(z-z_0)^m}{(z-z_0)^n} \frac{k(z)}{h(z)}$ does not have an ess. sing.

12 If $f(z)$ has a pole of order m at z_0, then $f(z) = \frac{1}{(z-z_0)^m} h(z)$ where $h(z)$ is analytic at z_0 and $h(z_0) \neq 0$. Then $f'(z) = \frac{-m}{(z-z_0)^{m+1}} h(z) + \frac{1}{(z-z_0)^m} h'(z)$

So $\dfrac{f'(z)}{f(z)} = \dfrac{-m\,(z-z_0)^m\,h(z)}{(z-z_0)^{m+1}\,h(z)} + \dfrac{(z-z_0)^m}{(z-z_0)^m}\dfrac{h'(z)}{h(z)} = \dfrac{-m}{z-z_0} + \dfrac{h'(z)}{h(z)}$.

Since $h(z_0) \neq 0$, $h'(z)/h(z)$ is analytic at $z = z_0$ and the residue is $-m$.

[13] If $f = \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k$ is the Laurent series for f near z_0 then



$$a_k = \frac{1}{2\pi i}\oint_{\gamma_\epsilon} f(z)\,(z-z_0)^{-k-1}$$ for any ccw circle γ_ϵ of radius ϵ about z_0.

So $$|a_k| \le \frac{1}{2\pi}\left(\max_{z\in\gamma_\epsilon} |f(z)|\,|z-z_0|^{-k-1}\right)\left(\text{length } \gamma_\epsilon\right)$$

$$\le \frac{M}{2\pi}\,\epsilon^{-k-1}\cdot 2\pi\epsilon = M\epsilon^{-k}$$

where $|f(z)| \le M$ for z near z_0. Since this is true for any small $\epsilon > 0$, it must be that $|a_k| = 0$ for $k < 0$.

[14] Suppose $|f(z) - c| \geq \delta > 0$ for z in a punctured nbd of z_0. Then $\left|\dfrac{1}{f(z)-c}\right| \le \dfrac{1}{\delta}$ is bounded in that punctured nbd. Thus

$\dfrac{1}{f(z)-c}$ must have a removable singularity so $\dfrac{1}{f(z)-c} = (z-z_0)^m h(z)$

where $h(z)$ is analytic near z_0, ($m \geq 0$) and $h(z_0) \neq 0$. Then $f(z) - c = \dfrac{1}{(z-z_0)^m}\dfrac{1}{h(z)}$

$f(z) = c + \dfrac{1}{(z-z_0)^m}\dfrac{1}{h(z)}$ has at most a pole at z_0.

[15] If $e^{f(z)}$ does not have an essential singularity at z_0, then either $e^{f(z)}$ has a removable singularity or a pole at z_0. If z_0 is removable then $|e^{f(z)}| \le M$ for all z near z_0. Thus $e^{\mathrm{Re}(z)} \le M$ so $\mathrm{Re} f(z) \le \log M$ for all z near z_0. But then the range of $f(z)$ omits a half plane and z_0 cannot be an essential singularity. Similarly, if z_0 is a pole then $\lim_{z\to z_0} e^{\mathrm{Re} f(z)} = \infty$ so $\lim_{z\to z_0} \mathrm{Re} f(z) = \infty$. Thus $\mathrm{Re} f(z) \geq 0$ for all z close enough to z_0 and thus again the range of f omits a half plane, and f cannot have an essential singularity at z_0.

6.1 p 313 1bdf 3ceg 5 7

$\boxed{1b}$ $\dfrac{z+1}{z^2-3z+2} = \dfrac{z+1}{(z-1)(z-2)}$ has simple poles at $z=1$, $z=2$. Using the $p(z_0)/q'(z_0)$ we have $p(z)/q'(z) = \dfrac{z+1}{2z-3}$ so $Res[\cdot;1] = \dfrac{2}{-1} = -2$, $Res[\cdot;2] = \dfrac{3}{4-3} = 3$

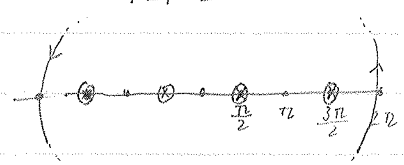
$\boxed{1d}$ $\left(\dfrac{z-1}{z+1}\right)^3$ has a pole of order 3 at $z=-1$. Method 1: $\left(\dfrac{z-1}{z+1}\right)^3 = \left(1-\dfrac{2}{z+1}\right)^3$

$= 1 - 3\left(\dfrac{2}{z+1}\right) + 3\left(\dfrac{2}{z+1}\right)^2 - \left(\dfrac{2}{z+1}\right)^3$ so $Res[\cdot;-1] = -6$. Method 2:

$Res[\cdot;-1] = \dfrac{1}{2}\lim_{z\to-1}\dfrac{d^2}{dz^2}(z-1)^3 = \dfrac{1}{2}\lim_{z\to-1} 3\cdot2\cdot(z-1) = -\dfrac{3\cdot2\cdot2}{2} = -6.$

$\boxed{1f}$ $\sin\left(\dfrac{1}{3z}\right)$ has an essential sing at $z=0$. Since $\sin\left(\dfrac{1}{3z}\right) = \dfrac{1}{3z} - \dfrac{1}{3!}\left(\dfrac{1}{3z}\right)^3 + \cdots$
$Res[\cdot;0] = \dfrac{1}{3}$.

$\boxed{3c}$ $I = \oint_{|z|=2\pi} \tan z \, dz$. $\tan z = \sin z/\cos z$ has singularities at $z = \frac{\pi}{2} + n\pi$



There are 4 singularities inside the contour: at $\pm\frac{\pi}{2}$ and $\pm\frac{3\pi}{2}$
They are all simple poles. Since $\dfrac{\sin z}{\cos'(z)} = \dfrac{\sin z}{-\sin z} = -1$

each residue has value -1. Thus $I = 2\pi i \cdot 4(-1) = -8\pi i$

$\boxed{3e}$ $I = \oint_{|z|=1} \dfrac{1}{z^2 \sin z}$ The only singularity inside contour is $z_0=0$, a pole of order 3. Method 1: $\dfrac{1}{z^2\sin z} = \dfrac{1}{z^2(z - z^3/3! + z^5/5! - \cdots)}$

$= \dfrac{1}{z^3(1-z^2/3!+z^4/5!-\cdots)} = \dfrac{1}{z^3}\left(1+(z^2/3!-z^4/5!+\cdots)+(z^2/3!-z^4/5!\cdots)^2 + \text{higher order}\right)$

By inspection, the residue is $\dfrac{1}{3!} = \dfrac{1}{6}$. Method 2: $Res[\cdot;0] = \lim_{z\to0}\dfrac{1}{2}\dfrac{d^2}{dz^2}\dfrac{z}{\sin z}$

$= \lim_{z\to0}\dfrac{1}{2}\dfrac{d}{dz}\dfrac{\sin z - \cos z\cdot z}{\sin z^2} = \lim_{z\to0}\dfrac{1}{2}\dfrac{(\cancel{\cos z}-\cancel{\cos z}+\sin z\cdot z)\sin^2 z - (\sin z - \cos z\cdot z)2\sin z\cos z}{\sin z^4}$

$= \lim_{z\to0}\dfrac{1}{2}\left(\dfrac{z}{\sin z} - \dfrac{(\sin z - \cos z\cdot z)\cdot 2\cos(z)}{(\sin z)^3}\right) = \dfrac{1}{2}\left(1 - \lim_{z\to0}\dfrac{(z - z^3/3! - (1-z^2/2)z + \text{higher order})\cdot 2}{\sin z^3}\right.$

$$= \frac{1}{2}\left(1 - \lim_{z\to 0}\left(\frac{z^3\left(\frac{1}{2}-\frac{1}{6}\right)+h.o.}{\sinh z^3}\right)\cdot 2\right) = \frac{1}{2}\left(1-\frac{2}{3}\right) = \frac{1}{6} \quad (\text{ugh.})$$

So $\quad I = 2\pi i \cdot \frac{1}{6} = \frac{i\pi}{3}$

$\boxed{3g}$ $I = \oint_{|z|=8} \frac{1}{z^2+z+1}\, dz \qquad$ Singularities at $z_\pm = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

$|z_\pm|^2 = 1$ so both singularites are inside contour. Easy way: Notice that $I = \mathrm{Res}\left[\frac{1}{z^2+z+1} ; \infty\right] = \lim_{z\to 0} \frac{z}{z^2+z+1} = 0$. Straightforward way: $\mathrm{Res}[\,\cdot\,; z_\pm]$

$= \frac{1}{2z_\pm + 1}$, so $\mathrm{Res}\left[\cdot, -\frac{1}{2}+i\frac{\sqrt{3}}{2}\right] = \frac{1}{-1+i\sqrt{3}+1} = -\frac{i}{\sqrt{3}}$ while $\mathrm{Res}\left[\cdot ; -\frac{1}{2}-i\frac{\sqrt{3}}{2}\right] = \frac{i}{\sqrt{3}}$

Thus $I = 2\pi i\left(-\frac{i}{\sqrt{3}} + \frac{i}{\sqrt{3}}\right) = 0$

$\boxed{5}$ A function $f(z)$ with a simple pole at z_0 has a Laurent expansion $f(z) = \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + \cdots$ with $a_{-1} \neq 0$. But $\mathrm{Res}[f; z_0] = a_{-1}$ so it can't be zero. If $f(z)$ has a pole of order 2 then the residue can be zero, eg for $f(z) = \frac{1}{(z-z_0)^2}$

$\boxed{7}$ $I = \oint_{|z|=1} e^{\frac{1}{z}} \sin\left(\frac{1}{z}\right) dz.$ $\boxed{\text{Ess. sing at } z=0}$ $\quad e^{\frac{1}{z}}\sin\left(\frac{1}{z}\right) = \left(1+\frac{1}{z}+\frac{1}{2z^2}+\frac{1}{3!\,z^3}\cdots\right)\left(\frac{1}{z}-\frac{1}{3!}\frac{1}{z^3}+\cdots\right)$

$= \frac{1}{z} +$ lower order terms. So $\mathrm{Res}\left[e^{\frac{1}{z}}\sin\left(\frac{1}{z}\right); 0\right] = 1$, $\quad I = 2\pi i$