Math 301 Hmk 2

6.2 (p 317) 2, 9, 10 (evaluate for all $n \in \mathbb{Z}$)

$\boxed{2}$ $$I = \int_0^{\pi} \frac{8\,d\theta}{5+2\cos\theta} = \frac{1}{2}\int_{-\pi}^{\pi} \frac{8\,d\theta}{5+2\cos\theta}\,d\theta \quad \text{(since cos is even)}$$

$$= \frac{1}{2i} \oint_{|z|=1} \frac{8}{5+\frac{2}{2}(z+\frac{1}{z})} \frac{dz}{z}$$

$$= \frac{8}{2i} \oint_{|z|=1} \frac{1}{z^2+5z+1}\,dz$$

Singularities at $z_{\pm} = -\frac{5}{2} \pm \frac{1}{2}\sqrt{25-4} = -\frac{5}{2} \pm \frac{1}{2}\sqrt{21}$

Since $z_+ z_-$ = the constant term $= 1$, $|z_+||z_-| = 1$ so one root is inside, one outside. z_+ is inside

$$\mathrm{Res}\left[\frac{1}{z^2+5z+1}, z_+\right] = \mathrm{Res}\left[\frac{1}{(z-z_+)(z-z_-)}; z_+\right] = \frac{1}{z_+-z_-}$$

$= \frac{1}{\sqrt{21}}$ so

$$I = \frac{8}{2i} \cdot 2\pi i \cdot \frac{1}{\sqrt{21}} = \frac{8\pi}{\sqrt{21}}$$

$\boxed{9.}$ $I = \int_0^{2\pi} (\cos\theta)^{2n} \, d\theta = \int_0^{2\pi} \frac{1}{2^{2n}} \left(z + \frac{1}{z} \right)^{2n} \frac{dz}{iz}$

$$= \frac{1}{i\, 2^{2n}} \oint_{|z|=1} \sum_{k=0}^{2n} \binom{2n}{k} z^k \left(\frac{1}{z}\right)^{2n-k} z^{-1} \, dz$$

$$= \frac{1}{i\, 2^{2n}} \sum_{k=0}^{2n} \binom{2n}{k} \oint_{|z|=1} z^{2k-2n-1} \, dz \qquad \text{only } k=n \text{ survives}$$

$$= \frac{2\pi}{2^{2n}} \binom{2n}{n} = \frac{\pi}{2^{2n-1}} \frac{(2n)!}{(n!)^2}$$

$\boxed{10.}$ $I = \int_0^{2\pi} e^{\cos\theta} \cos(n\theta - \sin\theta) \, d\theta = \int_0^{2\pi} \frac{1}{2} e^{\cos\theta} \left(e^{in\theta - i\sin\theta} + e^{-in\theta + i\sin\theta} \right) d\theta$

$$= \frac{1}{2} \int_0^{2\pi} \left[e^{\cos\theta - i\sin\theta} e^{in\theta} + e^{\cos\theta + is\theta} e^{-in\theta} \right] d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} \left[e^{e^{-i\theta}} e^{in\theta} + e^{e^{i\theta}} e^{-in\theta} \right] d\theta$$

$$= \frac{1}{2i} \oint_{|z|=1} \left(e^{\frac{1}{z}} z^n + e^z z^{-n} \right) \frac{dz}{z}$$

Singularity at $z=0$. To evaluate residue, expand

$$e^{\frac{1}{z}} z^{n-1} = z^{n-1} \sum_{k=0}^{\infty} \frac{1}{k!} z^{-k} = \sum_{k=0}^{\infty} \frac{1}{k!} z^{n-1-k}$$

$$\operatorname{Res}\left[e^{\frac{1}{z}} z^{n-1}; 0\right] = \begin{cases} 0 & \text{if } n < 0 \\ \frac{1}{n!} & \text{if } n \geq 0 \end{cases}$$

$$e^z z^{-n-1} = z^{-n-1} \sum_{k=0}^{\infty} \frac{1}{k!} z^k = \sum_{k=0}^{\infty} \frac{1}{k!} z^{-n-1+k}$$

$$\operatorname{Res}\left[e^z z^{-n-1}; 0\right] = \begin{cases} 0 & \text{if } n < 0 \\ \frac{1}{n!} & \text{if } n \geq 0 \end{cases}$$

$$I = \begin{cases} \frac{2\pi i}{2i} \cdot \left(\frac{1}{n!} + \frac{1}{n!} \right) = \frac{2\pi}{n!} & n \geq 0 \\ 0 & n < 0 \end{cases}$$

6.3 (p 325) 3, 10, 11

$\boxed{3}$ $$I = \int_0^\infty \frac{x^2+1}{x^4+1}\,dx = \frac{1}{2}\,pv\int_{-\infty}^{\infty}\frac{x^2+1}{x^4+1}\,dx$$

Singularities in UHP are $z = e^{i\pi/4}, e^{i3\pi/4}$ (simple poles) so

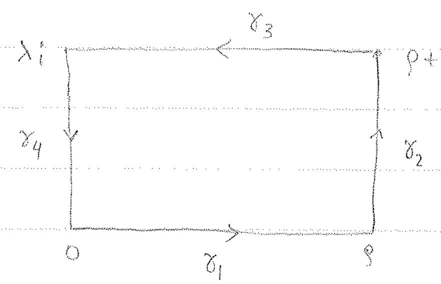
$$I = \frac{2\pi i}{2}\left\{ \text{Res}\left[\frac{z^2+1}{z^4+1}; e^{i\pi/4}\right] + \text{Res}\left[\frac{z^2+1}{z^4+1}; e^{i3\pi/4}\right]\right\}$$

$$= \pi i \left\{ \frac{e^{i\pi/2}+1}{4e^{i3\pi/4}} + \frac{e^{i6\pi/4}+1}{4e^{9\pi/4}} \right\}$$

$$= \frac{i\pi}{4}\left\{ e^{-i\pi/4} + e^{-i3\pi/4} + e^{-i3\pi/4} + e^{-i\pi 4} \right\}$$

$$= \frac{i\pi}{2}\left\{ \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} - \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} \right\} = \frac{\pi}{\sqrt{2}}$$

$\boxed{10}$



Since e^{-z^2} is analytic everywhere

$$\int_{\gamma_1+\gamma_2+\gamma_3+\gamma_4} e^{-z^2}\,dz = 0$$

$$\int_{\gamma_1} e^{-z^2}dz = \int_0^\rho e^{-x^2}dx \longrightarrow \frac{\sqrt{\pi}}{2} \quad \text{as } \rho\to\infty$$

$$\int_{\gamma_3} e^{-z^2}dz = -\int_0^\rho e^{-(x+i\lambda)^2} = -\int_0^\rho e^{-x^2-2i\lambda x+\lambda^2} = -e^{\lambda^2}\int_0^\rho e^{-x^2-2i\lambda x}dx$$

$$\int_{\gamma_2} e^{-z^2}dz = \int_0^\lambda e^{-(\rho+iy)^2}\,i\,dy = i\int_0^\lambda e^{-\rho^2-2iy\rho+y^2}dy$$

$$\left|\int_{\gamma_2} e^{-z^2}dz\right| \le \max_{y\in[0,\lambda]} e^{-\rho^2+y^2}\cdot\lambda \le e^{-\rho^2+\lambda^2}\cdot\lambda \longrightarrow 0 \quad \text{as } \rho\to\infty$$

$$\int_{\gamma_4} e^{-z^2}\,dz = -i\int_0^{\lambda} e^{y^2}\,dy$$

Therefore, taking the limit $\rho \to \infty$ we get.

$$\frac{\sqrt{\pi}}{2} + 0 - e^{\lambda^2}\int_0^{\infty} e^{-x^2-2i\lambda x}\,dx - i\int_0^{\lambda} e^{y^2}\,dy$$

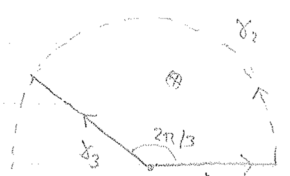
Take the real part: $\frac{\sqrt{\pi}}{2} = e^{\lambda^2}\int_0^{\infty} e^{-x^2}\cos(-2\lambda x)\,dx$ which $\Rightarrow$

$$\int_0^{\infty} e^{-x^2}\cos(2\lambda x)\,dx = \frac{e^{-\lambda^2}\sqrt{\pi}}{2}$$ since cos is even.

Take the imaginary part: $-e^{\lambda^2}\int_0^{\infty} e^{-x^2}\sin(-2\lambda x) = -\int_0^{\lambda} e^{y^2}\,dy$ which $\Rightarrow$

$$\int_0^{\infty} e^{-x^2}\sin(2\lambda x)\,dx = e^{-\lambda^2}\int_0^{\lambda} e^{y^2}\,dy.$$

$\boxed{\text{III}}$ $I = \int_0^{\infty}\frac{dx}{x^3+1}$. Following the hint, we use the contour



. Singularity inside contour is $e^{i\pi/3}$ so

$$\int_{\gamma_1} + \int_{\gamma_2} - \int_{\gamma_3} = 2\pi i\,\mathrm{Res}\left[\frac{1}{z^3+1}\,;\,e^{i\pi/3}\right] = 2\pi i\,\frac{1}{3e^{i2\pi/3}}$$

$$\int_{\gamma_1}^{\rho}\frac{1}{z^3+1}\,dz \longrightarrow I \text{ as } \rho\to\infty$$

$$\left|\int_{\gamma_2}\frac{1}{z^3+1}\,dz\right| \le \frac{1}{\rho^3+1}\cdot 2\pi/3\,\rho \longrightarrow 0 \text{ as } \rho\to\infty$$

$$\int_{\gamma_3}\frac{1}{z^3+1}\,dz = \int_0^{\rho}\frac{1}{(e^{i2\pi/3}x)^3+1}\cdot e^{i2\pi/3}x\,dx = e^{i\frac{2\pi}{3}}\int_0^{\rho}\frac{1}{x^3+1}\,dx$$

$$\to e^{i\frac{2\pi}{3}}\,I$$

So $\left(1-e^{i2\pi/3}\right) I = \frac{2\pi i}{3} e^{-i2\pi/3}$

$$I = \left(\frac{2\pi i}{3}\right) \frac{e^{-i2\pi/3}}{1-e^{i2\pi/3}} = \left(\frac{2\pi i}{3}\right) \frac{e^{-i\pi/3}}{e^{-i\pi/3}} \frac{e^{-i2\pi/3}}{1-e^{i2\pi/3}}$$

$$= \frac{2\pi i}{3} \frac{e^{-i\pi}}{e^{-i\pi/3} - e^{i\pi/3}} = \frac{\pi}{3} \frac{(-1)}{-\sin(\pi/3)} = \frac{\pi}{3} \frac{2}{\sqrt{3}} = \frac{2\pi\sqrt{3}}{9},$$