Math 301 Homework 4

filename: `hmk4.solutions.tex`

February 1, 2019

1. Show that $F(z) = \operatorname{Log}(-z) + i\pi$ is a branch of $\log(z)$ with branch cut on the positive real axis. Is it true that $F(z) = \operatorname{Log}_+(z)$? Here Log_+ denotes the branch of the log where the argument is chosen in $[0, 2\pi)$. (Hint: don't forget to check values right on the cut)

Solution: A function $F(z)$ is a branch of $\log(z) \iff F(z) \in \log(z)$ for every $z \iff \exp(F(z)) = z$ for all z. For the given F,

$$\exp(F(z)) = \exp(\operatorname{Log}(-z) + i\pi) = \exp(\operatorname{Log}(-z))\exp(i\pi) = (-z)(-1) = z$$

Thus $F(z)$ is a branch of the log.

Now we find the branch cut of F. The point z is on branch cut for F if $-z$ is on the cut for Log (the principal branch). This means that F has branch cut on $[0, \infty)$.

We now know that F and Log_+ have the same branch cut. Moreover $F(-1) = \operatorname{Log}(1) + i\pi = i\pi = \operatorname{Log}_+(-1)$. Thus F and Log_+ are branches with the same cut and the same value at a single point. This implies they are the equal ... except possibly right on the cut where we have to decide from which side to take a limit. And in fact $F(1) = \operatorname{Log}(-1) + \pi i = 2\pi i \neq \operatorname{Log}_+(1) = 0$.

Recall that for $z \in \mathbb{C}\backslash\{0\}$ and $\alpha \in \mathbb{C}$, the complex power z^α is defined to be $z^\alpha = \exp(\alpha \log(z))$ (as a multivalued function).

2. Show that $(zw)^\alpha = z^\alpha w^\alpha$ as sets. (The set on the right is $\{a \cdot b : a \in z^\alpha, b \in w^\alpha\}$)

Solution: We have $\xi \in (zw)^\alpha \iff$

$$\xi = \exp\left(\alpha\Big(\ln(|zw| + i\operatorname{Arg}(zw) + i2\pi k\Big)\right)$$

for some $k \in \mathbb{Z}$. On the other hand, $\xi \in z^\alpha w^\alpha \iff$

$$\begin{aligned}
\xi &= \exp\left(\alpha\Big(\ln(|z|) + i\operatorname{Arg}(z) + i2\pi l\Big)\right) \cdot \exp\left(\alpha\Big(\ln(|w| + i\operatorname{Arg}(w) + i2\pi m\Big)\right)\\
&= \exp\left(\alpha\Big(\ln(|z|) + i\operatorname{Arg}(z) + i2\pi l + \ln(|w| + i\operatorname{Arg}(w) + i2\pi m\Big)\right)\\
&= \exp\left(\alpha\Big(\ln(|zw|) + i\operatorname{Arg}(zw) + i2\pi(n + l + m)\Big)\right)
\end{aligned}$$

for some $l, m, n \in \mathbb{Z}$. Here the integer n is chosen so that $\operatorname{Arg}(z) + \operatorname{Arg}(w) = \operatorname{Arg}(zw) + i2\pi n$

Now the equality of the two sets follows from the fact that given $l, m, n \in \mathbb{Z}$, the sum $l + m + n$ is again in $\mathbb{Z}$, and conversely that for any choice of integer n, any integer k can be written as $k = n + l + m$

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3. Show that z^α

(a) is single valued if $\alpha \in \mathbb{Z}$,

(b) has q values if $\alpha = p/q$, where $p, q \in \mathbb{Z}$ with no common factors and $q > 0$. (I really should have said p and q are relatively prime.)

(c) has infinitely many values if α is irrational.

Solution:

(a) If $\alpha \in \mathbb{Z}$ then $\exp(2\pi i\alpha k) = 1$ for $k \in \mathbb{Z}$ so

$$\begin{aligned} z^\alpha &= \exp(\alpha \log(z)) \\ &= \{\exp(\alpha \operatorname{Log}(z) + i\alpha \operatorname{Arg}(z) + 2\pi i\alpha k) : k \in \mathbb{Z}\} \\ &= \{\exp(\alpha \operatorname{Log}(z) + 2\pi i\alpha \operatorname{Arg}(z)) \exp(2\pi i\alpha k) : k \in \mathbb{Z}\} \\ &= \{\exp(\alpha \operatorname{Log}(z) + 2\pi i\alpha \operatorname{Arg}(z))\}. \end{aligned}$$

So the set z^α contains a single value.

(b) From the calculation above, we see that the elements of the set z^α can be written as a fixed number $\exp(\alpha \operatorname{Log}(z) + 2\pi i\alpha \operatorname{Arg}(z))$ times the elements in the sequence $\{w_k : k \in \mathbb{Z}\}$, where $w_k = \exp(2\pi ipk/q)$. So we must determine how many distinct elements are in this sequence. The sequence is periodic with period q so we need only consider $\{w_k : k = 0, 1, \ldots, q-1\}$. We can always write $p = nq + r$ with $0 \le r < q$. Since we are assuming that q and p are relatively prime we must have $r \ne 0$.

Now w_k can also be written $w_k = \exp(2\pi irk/q)$. If there are repititions in the finite sequence above, that is, $w_k = w_l$ for $0 \le k, l < q-1$ with $k \ne l$ then $\exp(2\pi ir(k-l)/q) = 1$ which implies $r(k-l)/q = m \in \mathbb{Z}$ so that $r(k-l) = mq$. Since $r \ne 0$ and $k - l \ne 0$ we have that $m \ne 0$ so that q divides $r(k-l)$. But $k - l \in \{-q+1, \ldots, q-2, q-1\}$ so q cannot divide $k - l$. Similarly k cannot divide r since $0 \le r < q-1$. This is a contradiction. Thus there are no repititions.

(c) If w^α has only finitely many values then there must be repetitions in the sequence $\exp(2\pi i\alpha k)$ so that $\exp(2\pi i\alpha k) = \exp(2\pi i\alpha l)$ or $\exp(2\pi i\alpha(k-l)) = 1$ for some $k, l \in \mathbb{Z}$ with $k \ne l$. This implies $\alpha(k-l) = m \in \mathbb{Z}$ so that $\alpha = m/(k-l) \in \mathbb{Q}$.

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4. Identify the branch points of $f(z) = \log(z(z+1)/(z-1))$. (Don't forget to check $z = \infty$.) If we define a branch for $f(z)$ by choosing the principal branch of $\log(z)$, where are the branch cuts? (Note: this example illustrates that there may be a choice of branch cuts *not* obeying our "contractible loops" condition that still result in a single valued function.)

Solution:

Since $f(z) = \log(z) + \log(z+1) - \log(z-1)$ there are branch points at $z = 0, 1, -1$. In addition, when z goes around a large circle, the first two terms each increase by 2π while the last term decreases by 2π. So the total change in f is 2π and ∞ is also a branch point.

If we define $F(z) = \mathrm{Log}(z(z+1)/(z-1))$ then F is analytic near z unless $z(z+1)/(z-1)$ is on the branch cut for Log, i.e., $z(z+1)/(z-1) = -p$ for some $p \geq 0$. For all z except $z = 1$ (which will turn out to be an endpoint of the branch cut) this is equivalent to $z(z+1)+p(z-1) = z^2+(1+p)z-p = 0$. This has solutions $z+\pm = -\frac{1+p}{2} \pm \sqrt{\left(\frac{1+p}{2}\right)^2 + p}$. As p ranges from 0 to ∞, z_- ranges from -1 to $-\infty$ along the negative real axis while z_+ ranges from 0 to 1. (Here we can use that as $p \to \infty$, $z_+ = -\frac{1+p}{2} + -\frac{1+p}{2}\sqrt{1 + \left(\frac{2}{1+p}\right)^2 p} \sim -\frac{1+p}{2} + -\frac{1+p}{2}\left(2 + \frac{1}{2}\left(\frac{2}{1+p}\right)^2 p\right) = \frac{1}{2}\frac{2}{1+p}p \to 1$.) So we have shown that the branch cut for F is contained in $(-\infty, -1] \cup [0, 1]$. (To be complete we should verify that when we cross these intervals, the value of F really does jump, but for this problem it is okay if you omitted this step.)

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5. Find the branch points of $f(z) = (z^3 + z^2 - 6z)^{1/2}$. Define a branch $F(z)$ using the "range of angles" method that is continuous at $z = -1$ with $F(-1) = -\sqrt{6}$.

Solution:

Since $z^3 + z^2 - 6z = z(z+3)(z-2)$ there are branch points at $z = 0, -3, 2$. Also, since $f(z) = z^{3/2}\left(1 + \frac{1}{z} - \frac{6}{z^2}\right)^{1/2}$ we see that ∞ is also a branch point. To use the range of angles method we define r_i, φ_i for $i = 1, 2, 3$ by

$$z - 2 = r_1 e^{i\varphi_1}$$

$$z = r_2 e^{i\varphi_2}$$

$$z + 3 = r_3 e^{i\varphi_3}$$

Then $f(z) = (r_1 r_2 r_3)^{1/2} e^{i(\varphi_1+\varphi_2+\varphi_3)/2}$ and we can define a branch by choosing a range of angles for each φ_i. Take $\varphi_1 \in [0, 2\pi)$, $\varphi_2 \in [0, 2\pi)$ and $\varphi_3 \in (-\pi, \pi]$. Then for z close to -1, φ_1 is close to π, φ_2 is close to π, and φ_3 is close to 0, that is, they are away from the endpoints of their respective ranges. So there are no jumps near $z = -1$. The value of this branch at -1 is $(3 \cdot 1 \cdot 2)^{1/2} e^{i(\pi+\pi+0)/2} = \sqrt{6}e^{i\pi} = -\sqrt{6}$.

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6. Construct a branch $F(z)$ of $(z^2+1)^{1/2}$ that is

   (i) analytic inside the unit circle,

   (ii) analytic away from the imaginary axis,

   (iii) equals $\sqrt{x^2+1}$ for $x \in \mathbb{R}$.

   (iv) is continuous on the imaginary axis from the right.

Give an algorithm (i.e., a sequence of steps) that takes as input two real numbers x and y and computes $F(x+iy)$

Solution: Using the range of angles method we let $(z-i) = |z-i|e^{i\varphi_1}$ and $(z+i) = |z+i|e^{i\varphi_2}$ and define $F(z) = |z-i|^{1/2}|z+i|^{1/2}e^{i(\varphi_1+\varphi_2)/2} = \sqrt{|z^2+1|}e^{i(\varphi_1+\varphi_2)/2}$ where $\varphi_1 \in (-3\pi/2, \pi/2]$ and $\varphi_2 \in [-\pi/2, 3\pi/2)$. With this choice the cuts are $[i, i\infty)$ and $(-i\infty, -i]$ on the imaginary axis so (i) and (ii) hold. We have $\varphi_1 + \varphi_2 = 0$ when $z \in \mathbb{R}$ which implies (iii). Finally, the open and closed endpoints have been chosen to make (iv) true.
The algorithm could be something like

define the angles

```
phi1 = atan2(x, y-1);
phi2 = atan2(x, y+1);
```

the angles will (probably) be in $(-\pi, \pi]$ so we adjust

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if (phi1 > Pi/2) then phi1 = phi1-2*Pi end;
if (phi2 < -Pi/2) then phi2 = phi2+2*Pi end;
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the function output would then be

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F = sqrt(abs(z^2+1))*exp(i*(phi1+phi2)/2);
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