Problems from section 6.4

filename: `hmk6.4.tex`

January 28, 2020

6.4: 3 (not assigned)

$$\begin{aligned}\int_0^\infty \frac{\cos(x)}{(x^2+1)^2}dx &= \frac{1}{2}\int_{-\infty}^\infty \frac{\cos(x)}{(x^2+1)^2}dx \\ &= \frac{1}{2}\,\mathrm{Re}\int_{-\infty}^\infty \frac{e^{ix}}{(x^2+1)^2}dx \\ &= \frac{2\pi i}{2}\,\mathrm{Res}\left[\frac{e^{iz}}{(z^2+1)^2}; i\right].\end{aligned}$$

We have $\dfrac{e^{iz}}{(z^2+1)^2} = \dfrac{e^{iz}}{(z+i)^2(z-i)^2}$ so

$$\begin{aligned}\mathrm{Res}\left[\frac{e^{ix}}{(x^2+1)^2}; i\right] &= \lim_{z\to i}\frac{d}{dz}\frac{e^{iz}}{(z+i)^2} \\ &= \lim_{z\to i}\frac{ie^{iz}(z+i)^2 - 2(z+i)e^{iz}}{(z+i)^4} \\ &= \frac{ie^{-1}(-4) - 4ie^{-1}}{16} \\ &= \frac{-i}{2e}\end{aligned}$$

Thus $\int_0^\infty \dfrac{\cos(x)}{(x^2+1)^2}dx = \pi i\dfrac{-i}{2e} = \dfrac{\pi}{2e}$.

6.4: 6 (not assigned)

In this problem we close the contour in the lower half plane. This introduces a minus sign.

$$\begin{aligned}\int_{-\infty}^\infty \frac{e^{-2ix}}{x^2+4}dx &= -2\pi i\,\mathrm{Res}\left[\frac{e^{-2iz}}{z^2+4}; -2i\right] \\ &= -2\pi i\frac{e^{-4}}{-4i} \\ &= \frac{\pi}{e^4}\end{aligned}$$

6.4: 8

Since x^3 and $\sin(2x)$ are both odd functions of x, their product is even and so

$$\begin{aligned}\int_0^\infty \frac{x^3\sin(2x)}{(x^2+1)^2}dx &= \frac{1}{2}\int_{-\infty}^\infty \frac{x^3\sin(2x)}{(x^2+1)^2}dx \\ &= \frac{1}{2}\operatorname{Im}\left[\int_{-\infty}^\infty \frac{x^3e^{2ix}}{(x^2+1)^2}dx\right]\end{aligned}$$

We can evaluate this integral by closing the contour in the upper half plane. The only singularity in the upper half plane is at $z=i$. So

$$\int_{-\infty}^\infty \frac{x^3e^{2ix}}{(x^2+1)^2}dx = 2\pi i \operatorname{Res}\left[\frac{z^3e^{2iz}}{(z^2+1)^2}, z=i\right].$$

Since $(x^2+1)^2=(x+i)^2(x-i)^2$, there is a pole of order 2 at $z=i$ and

$$\begin{aligned}\operatorname{Res}\left[\frac{z^3e^{2iz}}{(z^2+1)^2}, z=i\right] &= \lim_{z\to i}\left(\frac{d}{dz}\right)(x-i)^2\frac{z^3e^{2iz}}{(z^2+1)^2} \\ &= \lim_{z\to i}\left(\frac{d}{dz}\right)\frac{z^3e^{2iz}}{(z+i)^2} \\ &= \lim_{z\to i}\frac{(3z^2e^{2iz}+2iz^3e^{2iz})(z+i)^2 - z^3e^{2iz}2(z+i)}{(z+i)^4} \\ &= 0\end{aligned}$$

This leads to

$$\int_0^\infty \frac{x^3\sin(2x)}{(x^2+1)^2}dx = 0$$

.

We can also find the residue by computing the Laurent series at $z=i$ The first few terms are

$$\frac{ie^{-2}}{4}(z-i)^{-2} + \frac{5ie^{-2}}{16} - \frac{11e^{-2}}{48}(z-i) + O(z-i)^2$$

confirming that i is a pole of order two with zero residue.

6.4: 9

Note that it is *not* true that $\frac{\cos(2x)}{x-3i} = \operatorname{Re}\frac{e^{2ix}}{x-3i}$. So in this problem we have to do two integrals:

$$\text{p.v.}\int_{-\infty}^\infty \frac{\cos(2x)}{x-3i}dx = \frac{1}{2}\int_{-\infty}^\infty \frac{e^{2ix}}{x-3i}dx + \frac{1}{2}\int_{-\infty}^\infty \frac{e^{-2ix}}{x-3i}dx$$

The second integral on the right is zero, because we close the contour in the lower half plane where $\frac{e^{-2iz}}{z-3i}$ is analytic. Thus we get

$$\frac{1}{2}\text{p.v.}\int_{-\infty}^\infty \frac{\cos(2x)}{x-3i}dx = \pi i \operatorname{Res}\left[\frac{e^{-2iz}}{z-3i}, 3i\right] = \pi i e^{-6}$$

6.4: 10

Suppose that $\mathrm{Im}(w) > 0$. Then there are no singularities in the lower half plane. This implies

$$p.v. \int_{-\infty}^{\infty} \frac{e^{-ix}}{x-w} dx = 0$$

because to compute this integral we must close the contour in the lower half plane and sum over the resonances there. On the other hand, still assuming that $\mathrm{Im}(w) > 0$,

$$p.v. \int_{-\infty}^{\infty} \frac{e^{ix}}{x-w} dx = 2\pi i \,\mathrm{Res}\left[e^{iz}/(z-w), z = w\right] = 2\pi i e^{iw};$$

Therefore

$$\begin{aligned} p.v. \int_{-\infty}^{\infty} \frac{\cos(x)}{x-w} dx &= p.v. \frac{1}{2} \int_{-\infty}^{\infty} \frac{(e^{ix} + e^{-ix})}{x-w} dx \\ &= p.v. \frac{1}{2} \int_{-\infty}^{\infty} \frac{e^{ix}}{x-w} dx + p.v. \frac{1}{2} \int_{-\infty}^{\infty} \frac{e^{-ix}}{x-w} dx \\ &= \pi i e^{iw} + 0 \end{aligned}$$

The case $\mathrm{Im}(w) < 0$ is similar.