Evaluation of some infinite sums

Proposition 0.1 *Let p and q be polynomials with $\deg(q) \geq \deg(p) + 2$ and let Q denote the (finite) set of roots of q. Define*

$$f(z) = \frac{p(z)\cot(\pi z)}{q(z)} = \frac{p(z)\cos(\pi z)}{q(z)\sin(\pi z)}$$

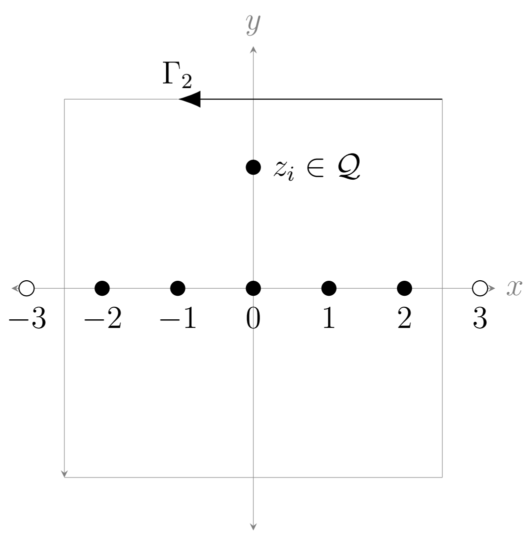
Then

$$\sum_{n \in \mathbb{Z} \backslash Q} \frac{p(n)}{q(n)} = -\pi \sum_{w \in Q} \operatorname{Res}[f; w]$$

Proof The function $f(z)$ has poles when $z \in \mathbb{Z} \cup Q$. The poles at $n \in \mathbb{Z} \backslash Q$ are simple and for these values of n

$$\operatorname{Res}[f; n] = \frac{p(n)}{\pi q(n)}.$$

Let Γ_n be a square with corners $(n + 1/2)(\pm 1 \pm i)$. For n large enough, Γ_n will enclose all the zeros of q.



The residue formula gives

$$\int_{\Gamma_n} f(z)dz = 2\pi i \left\{ \frac{1}{\pi} \sum_{k \in \mathbb{Z} \backslash Q, |k| \leq n} \frac{p(k)}{q(k)} + \sum_{w \in Q} \operatorname{Res}[f; w] \right\}$$

The idea is to show that the integral on the left tends to zero as $n \to \infty$. Standard bounds on polynomials give $|p(z)/q(z)| \leq C|z|^{-2}$ for $|z|$ large. This implies that for large n

$$\sup_{z \in \Gamma_n} \left| \frac{p(z)}{q(z)} \right| \leq \frac{C_1}{n^2},$$

Next we need to show that

$$\sup_{z\in\Gamma_n} |\cot(\pi z)| \le C_2.$$

Let $z = x + iy$ with $y > 0$. Then

$$|\cot(\pi z)| = \left|\frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} + e^{-i\pi z}}\right| \le \frac{|e^{i\pi z}| + |e^{-i\pi z}|}{|e^{-i\pi z}| - |e^{i\pi z}|} = \frac{e^{\pi y} + e^{-\pi y}}{e^{\pi y} - e^{-\pi y}} = \frac{1 + e^{-2\pi y}}{1 - e^{-2\pi y}}.$$

So if y is large enough to ensure $e^{-2y} \le 1/2$ then the right side is bounded by $(1+1/2)/(1-1/2) = 3$. Similarly, when $y < 0$ we get and

$$|\cot(\pi z)| \le \frac{1 + e^{2\pi y}}{1 - e^{2\pi y}}.$$

which is also bounded for y negative and large. This implies that $|\cot(\pi z)|$ is bounded by a constant independent of n on the top and bottom of Γ_n. We could also compute a bound for the left and right sides of Γ_n explicitly. Alternatively we can argue that $|\cot(\pi(n+1/2)+iy)|$ is a continuous function that tends to 1 for y tending to $\pm\infty$. Thus there must be a maximum value. But $|\cot(\pi z)|$ is a periodic function so the value of $|\cot(\pi(n+1/2)+iy)|$ is independent of n. This implies that we have a uniform bound for $|\cot(\pi z)|$ on the left and right sides of Γ_n. Altogether, we have $\sup_{z\in\Gamma_n} |\cot(\pi z)| \le C_2$ as required. Finally

$$\text{length}(\Gamma_n) \le C_3 n.$$

Thus

$$\left|\int_{\Gamma_n} f(z)dz\right| = \left|\int_{\Gamma_n} \frac{p(z)\cot(\pi z)}{q(z)} dz\right| \le \frac{C_1C_2C_3}{n} \to 0$$

as $n \to \infty$. This implies that

$$\lim_{n\to\infty} \left\{ \frac{1}{\pi} \sum_{k\in\mathbb{Z}\setminus Q, |k|\le n} \frac{p(k)}{q(k)} + \sum_{w\in Q} \text{Res}\,[f; w] = 0 \right\}$$

which gives the formula we wish to prove.

Example

$$\sum_{n=-\infty}^{\infty} \frac{1}{1+n^2} = -\pi \text{Res}\left[\frac{\cot(\pi z)}{1+z^2}; z = i\right] = -\pi \text{Res}\left[\frac{\cot(\pi z)}{1+z^2}; z = -i\right]$$
$$= -\pi \frac{\cot(-\pi i)}{-2i} = \pi \coth(\pi) \sim 3.153348095$$

Example

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{2} \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{n^2} = \frac{-1}{2\pi} \text{Res}\left[z^2 \cot(\pi z); z = 0\right] = \frac{1}{6}$$

Remark This method doesn't work for $\sum n^{-3}$ but we can get a formula for $\sum \frac{(-1)^n p(n)}{q(n)}$ using $f(z) = \frac{p(z)}{q(z)\sin(\pi z)}$.