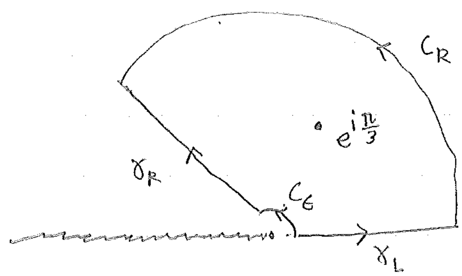
Ex: $I = \displaystyle\int_0^\infty \frac{\ln(x)}{1+x^3}\,dx \qquad f(z) = \frac{\mathrm{Log}(z)}{1+z^3}$



C_R bound:

$$|f(z)| = \frac{|\mathrm{Log}(z)|}{|1+z^3|} = \frac{|\ln|z| + i\,\mathrm{Arg}(z)|}{|1+z^3|} \le \frac{|\ln(R)+i\pi|}{R^3-1}$$

$$\underset{\downarrow\, I}{\int_{\gamma_L}} + \underset{\downarrow\, 0}{\int_{C_R}} - \int_{\gamma_R} - \underset{\downarrow\, 0}{\int_{C_\epsilon}} = 2\pi i \,\mathrm{Res}\left(\frac{\mathrm{Log}(z)}{1+z^3};\, e^{i\pi/3}\right) = \frac{2\pi i\,\mathrm{Log}(e^{i\pi/3})}{3e^{2i\pi/3}}$$

$$= \frac{2\pi i\; i\pi/3}{3e^{2\pi i/3}} = -\frac{2\pi^2}{9e^{2\pi i/3}}$$

γ_R: $z = e^{2\pi i/3} r \quad r \in [\epsilon, R] \quad dz = e^{2\pi i/3}\,dr \quad \mathrm{Log}(z) = \ln(r) + \frac{2\pi i}{3}$

$z^3 = r^3$!

$$\int_{\gamma_R} f\,dz = \int_\epsilon^R \frac{(\ln(r) + 2\pi i/3)}{1+r^3} e^{2\pi i/3}\,dr \to e^{2\pi i/3} I + e^{2\pi i/3}\frac{2\pi i}{3}\int_0^\infty \frac{1}{1+r^3}\,dr$$

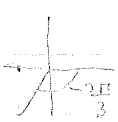
$$(1 - e^{2\pi i/3})\, I - e^{2\pi i/3}\frac{2\pi i}{3}\int_0^\infty \frac{1}{1+r^3}\,dr = -\frac{2\pi^2}{9}\frac{1}{e^{2\pi i/3}}$$

WHAT NOW? Take Re and Im parts ! we know I and $\int$ are real. Divide by $e^{2\pi i/3}$ first.

$$\left(\frac{1-e^{2\pi i/3}}{e^{2\pi i/3}}\right) I - \frac{2\pi i}{3}\int_0^\infty \frac{1}{1+r^3}\,dr = -\frac{2\pi^2}{9}\frac{1}{e^{4\pi i/3}};$$

$$\mathrm{Re}\left(e^{-2\pi i/3} - 1\right) I = -\frac{2\pi^2}{9}\,\mathrm{Re}\left(e^{2\pi i/3}\right) \to = \cos(+2\pi/3) = -\tfrac{1}{2}$$

$-\frac{1}{2}$, $\frac{\sqrt{3}}{2}$



$$-\frac{3}{2} I = \frac{-2\pi^2}{9}\cdot\frac{-1}{2} \qquad I = -\frac{2\pi^2}{27} \checkmark$$

Bonus integral $\quad (e^{-2\pi i/3} - 1) I - \frac{2\pi i}{3}\displaystyle\int_0^\infty \frac{1}{1+r^3}\,dr = -\frac{2\pi^2}{9}e^{+2\pi i/3}$

Im part: $\quad -\frac{\sqrt{3}}{2}\left(\frac{-2\pi^2}{27}\right) - \frac{2\pi}{3}\displaystyle\int_0^\infty \frac{1}{1+r^3}\,dr = \frac{-2\pi^2}{9}\left(\frac{\sqrt{3}}{2}\right) = -\frac{\sqrt{3}\pi^2}{9}$

$$-\frac{2\pi}{3}\int_0^\infty \frac{1}{1+r^3}\,dr = \sqrt{3}\,\pi^2\left(\frac{-1}{9}-\frac{1}{27}\right) = -\sqrt{3}\,\pi^2\,\frac{4}{27}$$

$$\int_0^\infty \frac{1}{1+r^3}\,dr = \frac{2\cdot 3\sqrt{3}\,\pi}{27} = \frac{2\sqrt{3}\,\pi}{9} \quad ✓$$

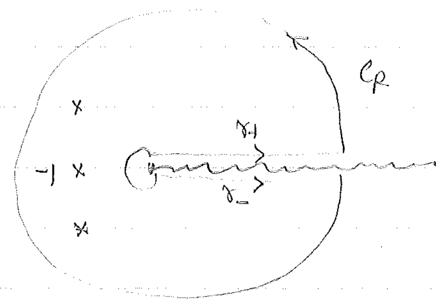
~~$\frac{2\sqrt{3}}{9}$ should be~~

Tricky example : Going for the bonus integral.

$$I = \int_0^\infty \frac{1}{(x+1)(x^2+2x-2)}\,dx$$

This will be the 'bonus' integral if we try to compute $\int_0^\infty \frac{\ln(x)}{(x+1)(x^2+2x-2)}\,dx$

Choose $f(z) = \dfrac{\mathrm{Log}_+(z)}{(z+1)(z^2+2z-2)}$



Sings at $-1, -1+i, -1-i$

$$\int_{\gamma_+} = \int_\epsilon^R \frac{\ln(x)}{(x+1)(x^2+2x-2)}\,dx$$

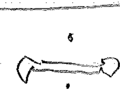
$$\int_{\gamma_-} = \int_\epsilon^R \frac{\ln(x)+i2\pi}{(x+1)(x^2+2x-2)}$$

$$\lim\left(\int_{\gamma_+} - \int_{\gamma_-}\right) = -2\pi i\int_0^\infty \frac{1}{(x+1)(x^2+2x-2)}\,dx = 2\pi i \sum_{z\in\{-1,-1+i,-1-i\}} \mathrm{Res}\left[\frac{\mathrm{Log}_+(z)}{(z+1)(z^2+2z-2)}\right]$$

$$= \underset{\substack{\uparrow \\ \text{work!}}}{\cdots} = 2\pi i\left(-\ln(\sqrt{2})\right)$$

ln integral cancels!

$$I = \ln(\sqrt{2}).$$

Next time : examples like $\int_{-1}^{1} \frac{(1-x^2)^{\frac{1}{2}}}{x^2+1}\,dx$  Dogbone contour