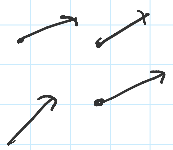
M301 Ideal Fluid flow

March 13, 2020 10:00 AM

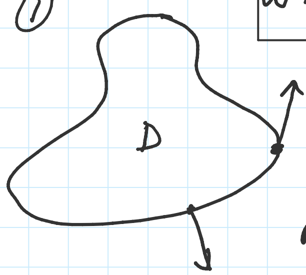
Steady 2-d fluid flow described by assigning a velocity $v = (v_1, v_2)$ to each point.

Assume ① incompressible
② irrotational



$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

①



$\boxed{\text{What does this say about } \vec{v}\text{?}}$

$r(t) = (x(t), y(t))$ follows ∂D $t \in [0,1]$

$\dot{r}(t)$ tangent $= (\dot{x}(t), \dot{y}(t))$

$$\vec{n}(t) = \frac{(-\dot{y}(t), \dot{x}(t))}{\sqrt{\dot{x}^2(t) + \dot{y}^2(t)}}$$ unit normal

① $\Leftrightarrow$ Total flux in/out of D is 0.

$$\int_{\partial D} \vec{n} \cdot \vec{v}\, ds = 0 \qquad ds = \sqrt{\dot{x}^2 + \dot{y}^2}\, dt$$

$$\int_0^1 \frac{-\dot{y} v_1 + \dot{x} v_2}{\sqrt{\dot{x}^2 + \dot{y}^2}} \sqrt{\dot{x}^2 + \dot{y}^2}\, dt$$

$$= \int_{\partial D} -v_1\, dy + v_2\, dx$$

$$= -\int_D \left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} \right) dx\, dy$$

True for any D $\Rightarrow$ $\boxed{\dfrac{\partial v_1}{\partial x} + \dfrac{\partial v_2}{\partial y} = 0}$ ①

(2) Circulation around any closed contour is 0

$$\int_{\partial D} v_1\,dx + v_2\,dy = 0 \implies \iint_D \left(-\frac{\partial v_1}{\partial y} + \frac{\partial v_2}{\partial x}\right) = 0$$

$$\Rightarrow \boxed{\frac{\partial v_1}{\partial y} = \frac{\partial v_2}{\partial x}} \quad (2)$$

Defining the complex velocity potential

$$(2) \implies \vec{V} = \nabla \Phi$$

$$(1) \implies \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$$

Let Ψ be the harmonic conjugate

$\Omega(x+iy) = \Phi(x,y) + i\Psi(x,y)$ is analytic $\to$ called the <u>complex velocity potential</u>

$$\Omega'(z) = \Phi_x + i\Psi_x = \Phi_x - i\Phi_y$$

$$= v_1 - i v_2$$

So $\overline{\Omega'(z)} = v_1 + i v_2$

speed at (x,y) is $|\Omega'(x+iy)|$

speed at (x, y) is $|\Omega'(x+iy)|$

<u>Stagnation points</u> where $\Omega'(z) = 0$

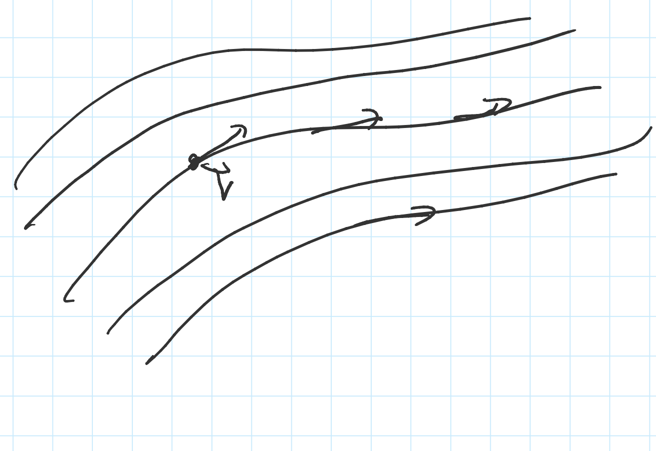
<u>Defn</u> $\Psi(x,y) = \mathrm{Im}(\Omega(x+iy))$ is called the <u>stream fcn</u>

We know that level curves of Ψ and of Φ are perpendicular

$\nabla\Phi$ is also $\perp$ to level curves of Φ

$\Rightarrow$ The level curves of Ψ are tangent to $\vec{v}$

$\Psi = c$



Example : steady flow with $\vec{V} = v_0(\cos(\alpha), \sin(\alpha))$

$$\overline{\Omega'(z)} = v_0 e^{i\alpha} \qquad v_0 \in \mathbb{R}$$

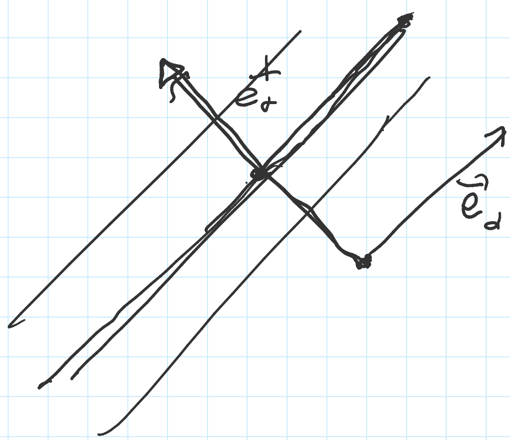
$$\Omega'(z) = v_0 e^{-i\alpha}$$

$$\Omega(z) = v_0 e^{-i\alpha} z + c$$

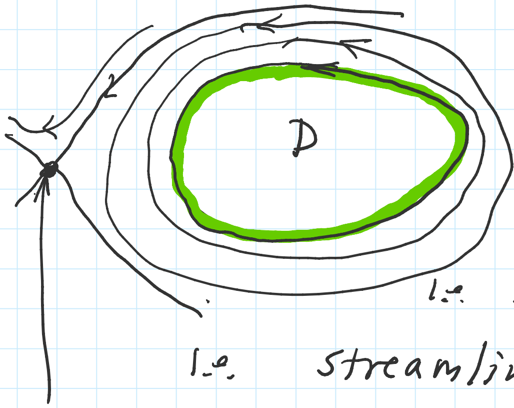
ignore

Let $e_\alpha = \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$

$$\Phi = v_0 \vec{e}_\alpha \cdot \vec{x} \qquad \Psi = v_0 \vec{e}_\alpha^{\perp} \cdot \vec{x}$$



Fluid flow around a body



Fluid flow around a body D is described by a velocity potential $\Omega(z)$ with $\mathrm{Im}(\Omega(z)) = C$ for $z \in \partial D$.

i.e. $\Psi(z) = C$ on the boundary

i.e. streamline flow follows the boundary

z is a stagnation pt $\iff \Omega'(z) = 0$.

Example: Fluid flow around a cylinder

Consider Joukowski map

$$\Omega(z) = V_0\left(z + \frac{a^2}{z}\right) \qquad V_0 > 0 \quad a > 0$$

$z = x+iy$

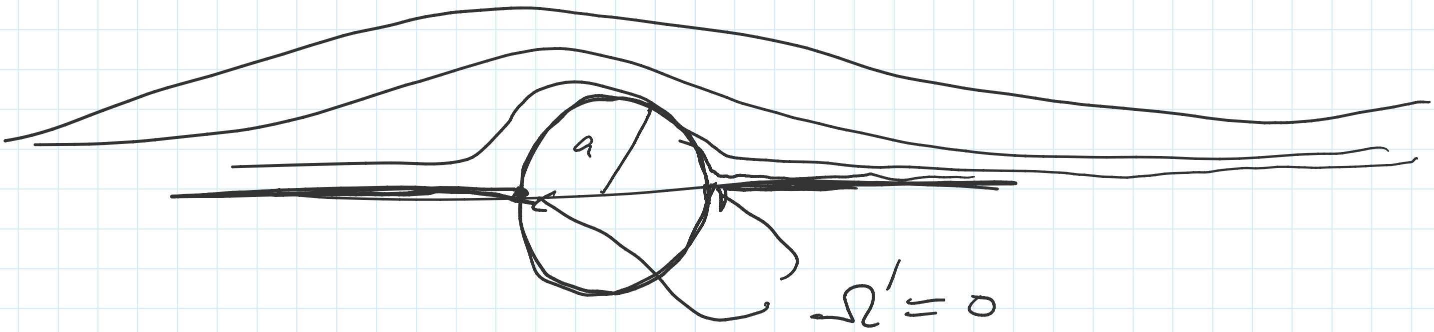
$$\mathrm{Im}\,\Omega(z) = V_0 y + \mathrm{Im}\left(\frac{a^2(x-iy)}{x^2+y^2}\right)$$

$$= V_0\left(y - \frac{a^2 y}{x^2+y^2}\right)$$

$z = re^{i\theta} \qquad y = r\sin(\theta)$

$$= V_0 \left(r\sin\theta - \frac{a^2 r \sin\theta}{r^2} \right)$$

$$V_0 \sin\theta \left(r - \frac{a^2}{r} \right) = \operatorname{Im}(\Omega) = \Psi$$



Stagnation

other level curves $\quad V_0 \sin\theta \left(r - \frac{a^2}{r} \right) = c$

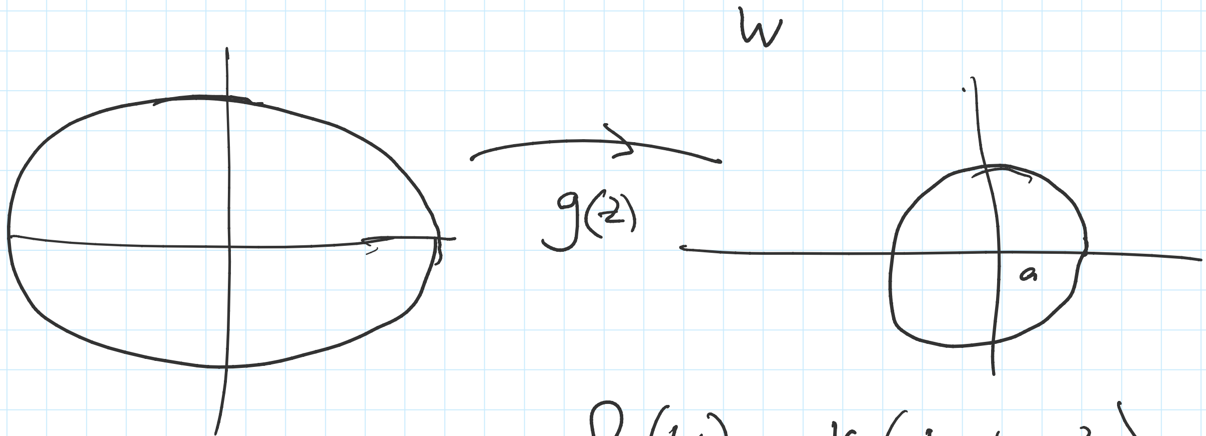
$$\Downarrow$$

$$r = \frac{c}{2\sin\theta} + \sqrt{\frac{c^2}{4\sin^2\theta} + a^2}$$

$$\boxed{\text{Example : Fluid flow around an ellipse}}$$

Need to find a conformal map that maps exterior of an ellipse to exterior of circle,

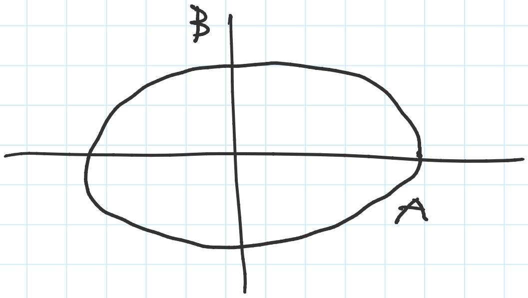
w

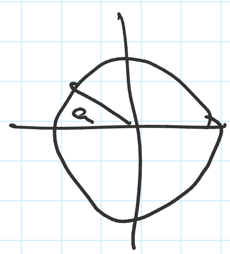


$$\Omega_0(w) = V_0\left(w + \frac{a^2}{w}\right)$$

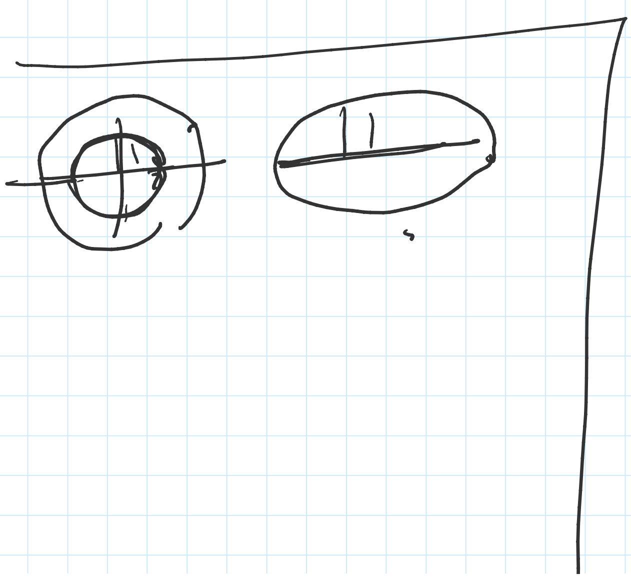
If g is the conf. map mapping ∂ into ∂

$$\Omega(z) = \Omega_0(g(z))$$



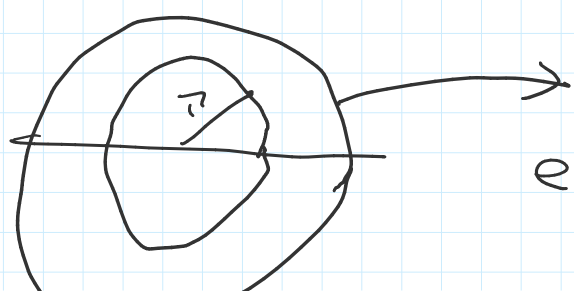


$$\Omega_0(w) = V_0\left(w + \frac{a^2}{w}\right)$$

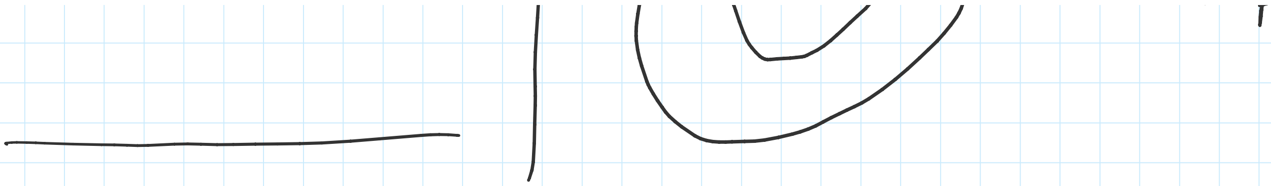


Fact: $\Omega_r(w) = V_0\left(w + \frac{r^2}{w}\right)$

maps



ellipse.



$$a = \frac{A+B}{2} \qquad \tau^2 = \frac{A^2 - B^2}{4}.$$

The inverse map:

$$J = w + \frac{\tau^2}{w} = z \qquad w^2 - wz + \tau^2 = 0$$

$$w = \frac{z \pm \sqrt{z^2 - 4\tau^2}}{2}$$

Choose a branch