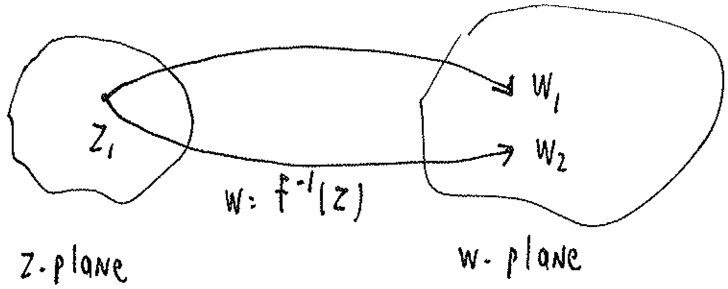
MULTI-VALUED FUNCTIONS

THERE ARE MANY FUNCTIONS WHOSE INVERSE FUNCTION IS MULTI-VALUED FOR INSTANCE

$$z = e^{w}, \quad z = w^{2}, \quad z = \cos w, \quad z = \sin w.$$

FOR EACH OF THESE FUNCTIONS, A GIVEN VALUE OF Z CORRESPONDS TO MORE THAN ONE VALUE OF W



- $w = f^{-1}(z)$ is multi-valued
- $z = f(w)$ is single valued. GIVEN A w, THERE IS A UNIQUE VALUE OF Z

<u>GOALS</u> (i) DETERMINE ALL POSSIBLE VALUES OF THE INVERSE FUNCTION W.

(ii) CONSTRUCT AN INVERSE FUNCTION THAT IS SINGLE VALUED IN SOME REGION OF COMPLEX PLANE.

LOGARITHM FUNCTION

DEFINE THE INVERSE FUNCTION FOR $z = e^{w}$.

FOR A GIVEN $z = re^{i\varphi}$ WITH $\varphi = \mathrm{ARG}(z)$ WE WRITE $w = u + iv$ WHERE u, v TO BE FOUND.

SO $$re^{i\varphi} = e^{u+iv}$$

TAKING THE MODULUS WE GET $|re^{i\varphi}| = |e^{u+iv}| \rightarrow r = e^{u}$ SO $u = \ln r$.

THEN $v = \varphi + 2k\pi \qquad k = 0, \pm 1, \pm 2, \ldots$

HENCE $$w = \ln r + i\left[\mathrm{Arg}(z) + 2k\pi\right] \qquad k = 0, \pm 1, \pm 2.$$

THIS IS $$w = \ln|z| + i\left[\mathrm{Arg}(z) + 2k\pi\right] \qquad -\pi < \mathrm{ARG}(z) \leq \pi$$
$$k = 0, \pm 1, \pm 2, \ldots$$

WE DEFINE THE <u>MULTI-VALUED</u> $\log z$ BY

$$w = \log z \equiv \ln|z| + i\left(\mathrm{ARG}(z) + 2k\pi\right) \qquad k = 0, \pm 1, \pm 2, \ldots$$

IT GIVES ALL THE SOLUTIONS w TO $z = e^{w}$.

EQUIVALENTLY, WE CAN DEFINE IT AS

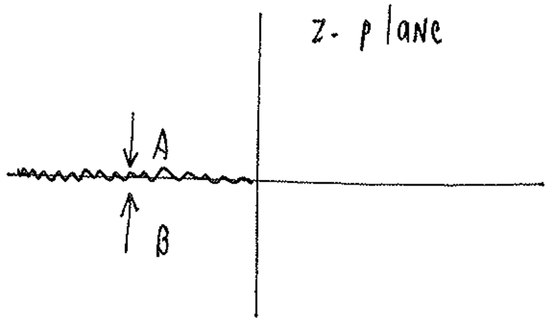
$$w = \log z = \ln|z| + i(\arg z)$$

SINCE $\arg z$ IS multi-valued.

WE DEFINE THE <u>PRINCIPAL VALUE</u> OF $\log z$ BY

$$w \;\; \mathrm{LOG}\, z \equiv \ln|z| + i\,\mathrm{Arg}(z).$$

SINCE $-\pi < \mathrm{ARG}(z) \leq \pi$ WE NOTICE THAT w IS NOT CONTINUOUS AT ANY POINT ON THE NEGATIVE REAL AXIS.



A: AS $z = x + iy$ WITH $y \to 0^{+}$ WITH $x < 0$ THEN

$$\mathrm{LOG}\, z \to \ln|x| + i\pi.$$

B: AS $z = x + iy$ WITH $y \to 0^{-}$ WITH $x < 0$ THEN

$$\mathrm{LOG}\, z \to \ln|x| - i\pi.$$

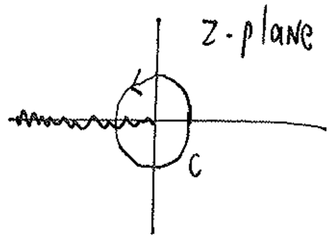
THUS, $\mathrm{LOG}\, z$ IS DISCONTINUOUS ON NEGATIVE REAL AXIS.

<u>REMARK</u> (i) $\mathrm{LOG}\, z$ IS CALLED A "BRANCH" OF THE MULTI-VALUED FUNCTION $\log z$

(ii) $\mathrm{LOG}\, z$ IS CONTINUOUS IN THE CUT PLANE $\mathbb{C} \setminus (-\infty, 0]$.

(iii) THE POINT $z = 0$ IS CALLED A "BRANCH POINT" OF $\mathrm{LOG}\, z$, SINCE IF WE ENCIRCLE $z = 0$ BY

A CLOSED CONTOUR THEN LOG Z CHANGES BY AN AMOUNT PROPORTIONAL TO $2\pi i$



CHANGE IN LOG Z AROUND PATH C IS $2\pi i$.

- IN THE CUT PLANE LOG Z IS ANALYTIC AND $\frac{d}{dz} \operatorname{LOG} z = \frac{1}{z}$.

PROOF $F(z) = \operatorname{LOG} z = \frac{1}{2} \ln(x^2 + y^2) + i \tan^{-1}(y/x)$

WITH $-\pi < \tan^{-1}(y/x) \le \pi$.

THEN $u = \frac{1}{2} \ln(x^2+y^2)$ $\quad v = \tan^{-1}(y/x)$

$$u_x = \frac{x}{x^2+y^2} \qquad v_x = \frac{-y/x^2}{1+y^2/x^2} = \frac{-y}{x^2+y^2}.$$

$$u_y = \frac{y}{x^2+y^2} \qquad v_y = \frac{1/x}{1+y^2/x^2} = \frac{x}{x^2+y^2}.$$

SO $u_x = v_y$, $u_y = -v_x$ PROVIDED $(x,y) \neq (0,0)$.

AND $F'(z) = u_x + i v_x = \frac{x}{x^2+y^2} - \frac{iy}{(x^2+y^2)} = \frac{\bar{z}}{|z|^2} = \frac{\bar{z}}{z\bar{z}} = \frac{1}{z}$.

SO $F'(z) = 1/z$

EXAMPLE CALCULATE THE FOLLOWING:

(i) $\log(2i)$ (ii) $\operatorname{LOG}(1 + i\sqrt{3})$ (iii) $\operatorname{LOG}(-i)$

SOLUTION

(i) $\log(2i)$: let $z = 2i$. THEN $\operatorname{Arg}(z) = \pi/2$.

SO $\log(2i) = \ln 2 + i\left[\pi/2 + 2k\pi\right]$ $\quad k = 0, \pm 1, \pm 2, \ldots$

(ii) $1 + i\sqrt{3} = 2e^{\pi i/3}$. $\quad \text{ARG}(1+i\sqrt{3}) = \pi/3$

SO $\text{LOG}(1+i\sqrt{3}) = \ln 2 + i\left[\pi/3 + 2k\pi\right] \quad k = 0, \pm 1, \pm 2, \ldots$

(iii) $\text{LOG}(-i)$. LET $z = -i$. $\quad |z| = 1, \ \text{Arg}\, z = -\pi/2$.

SO $\text{LOG}(-i) = \ln 1 + i\left(-\pi/2\right)$. $\quad$ SO $\text{LOG}(-i) = -i\pi/2$.

ONE MUST BE CAREFUL WITH IDENTITIES INVOLVING $\text{LOG}\, z$, $\log z$.

THE FOLLOWING RESULTS, AS SHOWN IN HW, HOLD

(i) $\text{LOG}(z_1 z_2) \neq \text{LOG}(z_1) + \text{LOG}(z_2)$

(ii) $\log(z_1 z_2) = \log z_1 + \log z_2$

$\log(z_1/z_2) = \log z_1 - \log z_2 \quad (z_2 \neq 0)$

THE COUNTABLY INFINITE set OF values OF the left-hand-side AND right-hand-side are same

(iii) $\log(e^z) \neq z$

(iv) $\text{LOG}(e^z) = z$ IF AND ONLY IF $-\pi < \text{IM}(z) \leq \pi$.

(v) $\log z = -\log(1/z)$

(vi) $z = e^{\log z}$

(vii) $\log(z^{1/n}) = \frac{1}{n} \log z \quad$ n: positive integer

(viii) $\log(z^n) \neq n \log z$ IN general (n: positive integer).

WE NOW GIVE A PROOF FOR A FEW OF THESE. YOU ARE ASKED TO PROVE THE OTHERS IN THE HOMEWORK.

PROOF (vi) $\log z = \ln|z| + i[\operatorname{ARG} z + 2k\pi]$. (M5)

SO $$e^{\log z} = e^{\ln|z| + i[\operatorname{ARG} z + 2k\pi]} = |z| e^{i\operatorname{ARG}(z)} = z.$$

PROOF (viii) WE WILL SHOW THAT THE SETS OF VALUES OF

$\log(z^n)$ AND $n \log z$ DO NOT COINCIDE

LET $z = re^{i\varphi}$ WITH $\varphi = \operatorname{ARG} z$. WE GET $z^n = r^n e^{in\varphi}$

THEN $\log(z^n) = \ln(r^n) + i[n\varphi + 2k\pi]$, $k = 0, \pm1, \pm2, ..$

• $\log(z^n) = n \ln r + i(n\varphi + 2k\pi)$

BUT • $n \log z = n[\ln r + i(\varphi + 2m\pi)] = n \ln r + i[n\varphi + 2mn\pi]$.

COMPARING THESE TWO SETS IS EQUIVALENT TO COMPARING

$$\{2k\pi\} \quad \text{AND} \quad \{2mn\pi\}$$

$k = 0, \pm1, \pm2..$ $\quad$ $m = 0, \pm1, \pm2, ..$ $\quad$ ($n > 0$ integer FIXED)

THESE ARE NOT IN GENERAL THE SAME. IN PARTICULAR IF $n = 2$ THEN

$$\{2k\pi\} = \{0, \pm2\pi, \pm4\pi, ...\}$$
$$\{2mn\pi\} = \{4m\pi\} = \{0, \pm4\pi, \pm8\pi, ...\}$$
NOT SAME.

PROOF (iii) SHOW $\log(e^z) \neq z$. NOTICE LEFT-HAND-SIDE IS MULTIVALUED, BUT RHS IS SINGLE VALUED.

PUT $z = x + iy$. $\quad e^z = e^x e^{iy}$ $\quad$ $\operatorname{ARG}(e^z) = \operatorname{ARG}(e^{iy}) = y$. IF $-\pi < y \leq \pi$

SO $\log(e^z) = \ln|e^z| + i[y + 2k\pi i] = \ln e^x + i(y + 2k\pi i)$
IF $-\pi < y \leq \pi$.

THUS $\log(e^z) = z + i(2k\pi + y)$, IF $-\pi < \operatorname{IM} z \leq \pi$.

PROOF (vii) show that the sets $\log(z^{1/n})$ AND $\frac{1}{n}\log z$ ARE THE SAME WHERE n = positive integer.

write $z = re^{i\varphi}$ WITH $\varphi = \mathrm{ARG}(z)$. THEN $z^{1/n} = r^{1/n} e^{i(\varphi + 2k\pi)/n}$ $k = 0, .., n-1$.

THUS $\log(z^{1/n}) = \frac{1}{n}\ln r + i\left[\frac{\varphi + 2k\pi}{n} + 2p\pi\right]$ $k = 0, 1, .., n-1$; $p = 0, \pm1, \pm2, ..$

NOW $\frac{1}{n}\log z = \frac{1}{n}\ln r + i\left[\frac{\varphi}{n} + \frac{2q\pi}{n}\right]$ $q = 0, \pm1, \pm2, ..$

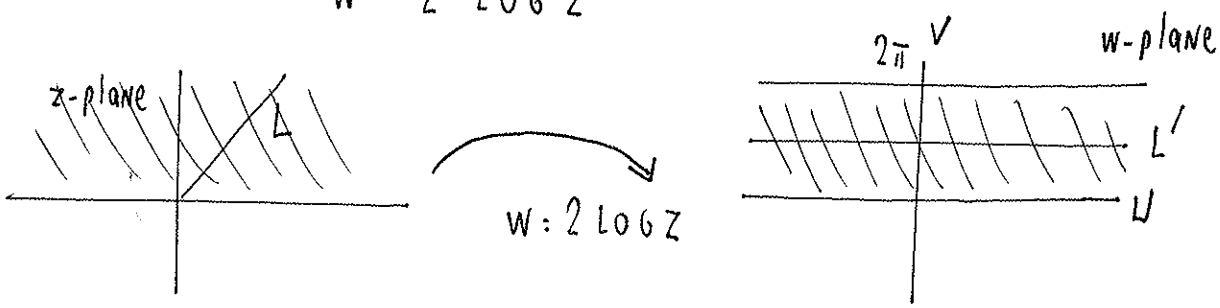
THE SET OF VALUES OF $\log(z^{1/n})$ AND $\frac{1}{n}\log z$ ARE THE SAME IF THE TWO SETS $\{k + pn\}$ $k = 0, .., n-1$; $p = 0, \pm1, \pm2$ coincide with the set $\{q\}$ $q = 0, \pm1, \pm2, ..$

THIS IS TRUE
- FOR ANY k AND p WE GET A q
- DIVIDING q BY n WE GET AN INTEGER AND A REMAINDER k IN $\{0, .., n-1\}$.

EXAMPLE OF MAPPING INVOLVING LOG Z

EX: FIND THE IMAGE OF $S = \{z \mid \mathrm{IM}\, z \geq 0\}$ UNDER THE MAPPING

$$W = 2 \,\mathrm{LOG}\, Z$$



TO PARAMETRIZE Z-PLANE LET $z = re^{i\varphi}$. THEN

$$W = 2\left[\ln r + i\varphi\right] \quad \text{WITH} \quad 0 \leq \varphi \leq \pi. \quad \text{WRITE} \quad W = U + iV.$$

HENCE $U = 2\ln r$ $\quad V = 2\varphi$

- FIX A RAY WITH φ FIXED (line L in Z-PLANE above). THEN SINCE $0 < r < \infty$ WE GET U IN $(-\infty, \infty)$ AND V = FIXED. THE IMAGE LINE L' IS SHOWN IN W-PLANE ABOVE
- SINCE $0 \leq \varphi \leq \pi$ WE GET U IN $(-\infty, \infty)$ AND V IN $(0, 2\pi)$

HENCE $S' = \{w \mid 0 \leq \mathrm{IM}\, w \leq 2\pi\}$.

EXPONENTS

IF α IS A COMPLEX NUMBER AND $z \neq 0$ THEN WE DEFINE $z^{\alpha} = e^{\alpha \log z}$ (multi-valued).

THUS, $z^{\alpha} = e^{\alpha[\ln|z| + i(\mathrm{ARG}\, z + 2K\pi)]}$ $K = 0, \pm 1, \pm 2, \ldots$

THIS YIELDS THAT

$$z^{\alpha} = |z|^{\alpha} e^{i[\alpha \mathrm{ARG}(z) + 2K\pi\alpha]}, \quad K = 0, \pm 1, \pm 2, \ldots$$

- THERE WILL BE A FINITE NUMBER OF VALUES OF z^{α} ONLY IF α IS THE RATIO OF TWO INTEGERS (I.E. IS RATIONAL). IN SUCH A CASE αK = integer FOR some K.

THE PRINCIPAL VALUE OF z^{α} IS DEFINED BY

$$z^{\alpha} = e^{\alpha \mathrm{LOG}(z)} = e^{\alpha[\ln|z| + i\mathrm{ARG}(z)]}$$

SINCE $\mathrm{LOG}(z)$ IS ANALYTIC IN THE SLIT DOMAIN $C \setminus (-\infty, 0)$ AND e^{w} IS ANALYTIC, THEN z^{α} IS ANALYTIC IN $C \setminus (-\infty, 0]$

AND $\frac{d}{dz} z^{\alpha} = \alpha z^{\alpha} \frac{1}{z} = \alpha z^{\alpha - 1}$ IN $C \setminus (-\infty, 0]$.

EXAMPLE FIND ALL SOLUTIONS TO $z^{1+i} = 4$. WE WRITE $z^{1+i} = e^{(1+i)\log z} = e^{\ln 4}$. THUS $(1+i)\log z = \ln 4 + 2\pi n i$, $n = 0, \pm 1, \pm 2$

THUS $2 \log z = (1-i)[\ln 4 + 2\pi n i] \Rightarrow \log z = (1-i)[\ln 2 + \pi n i]$.

HENCE $\log z = \ln 2 + \pi n + i(\pi n - \ln 2)$. NOW EXPONENTIATING

GIVES $z = 2e^{\pi n + i[n\pi - \ln 2]} = 2e^{\pi n}(-1)^{n}[\cos(\ln 2) - i\sin(\ln 2)]$

SINCE $e^{in\pi} = (-1)^{n}$.

DEFINITION

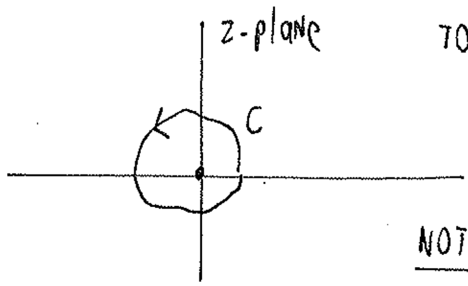
$\hat{f}(z)$ IS A BRANCH OF THE MULTI-VALUED FUNCTION $f(z)$ IN A DOMAIN D IF $\hat{f}(z)$ IS SINGLE-VALUED AND CONTINUOUS IN D AND HAS THE PROPERTY THAT FOR EACH z IN D THE VALUE $\hat{f}(z)$ IS ONE OF THE VALUES OF $f(z)$.

TO CONSTRUCT $\hat{f}(z)$ WE INTRODUCE A CURVE EMANATING FROM A POINT (CALLED THE BRANCH POINT) TO ENSURE THAT $\hat{f}(z)$ IS SINGLE-VALUED IN THE CUT PLANE. A BRANCH POINT IS A POINT FOR WHICH IF WE ENCIRCLE IT WITH AN ARBITRARY SUFFICIENTLY SMALL CURVE THE FUNCTION $f(z)$ CHANGES DISCONTINUOUSLY.

ALTHOUGH A DEEPER UNDERSTANDING OF THESE ISSUES REQUIRES MORE ADVANCED TOPICS (i.e. RIEMANN SURFACES), WE CAN STILL ILLUSTRATE THE IDEAS WITH SOME EXAMPLES.

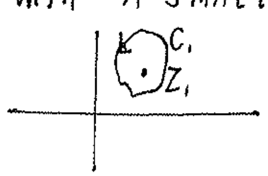
EXAMPLE 1 LET $f(z) = \log z$. (multi-valued).

THE POINT $z = 0$ IS A BRANCH POINT SINCE IF WE TAKE A PATH C AS SHOWN BELOW, THEN $\log z$ DOES NOT RETURN TO ITS ORIGINAL VALUE. THE CHANGE $[\log z]_C$ IS



$$[\log z]_C = 2\pi i .$$

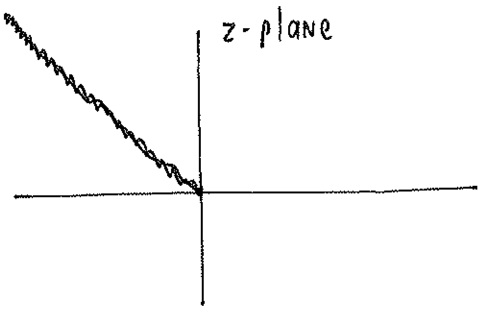
NOTE: IF WE ENCIRCLE ANY OTHER POINT $z_1 \neq 0$ WITH A SMALL CLOSED CURVE C_1 (AS SHOWN)



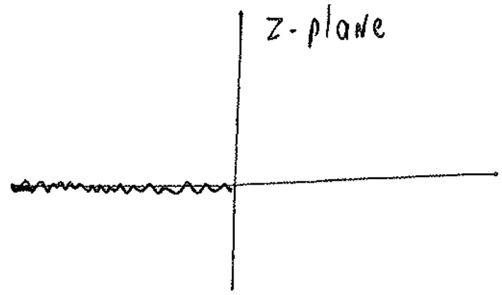
$[\log z]_{C_1} = 0$ THUS $z_1 \neq 0$ IS NOT A BRANCH POINT.

WE MUST INSERT A CURVE, CALLED THE BRANCH CUT, TO PREVENT COMPLETE CIRCUITS ABOUT THE BRANCH POINT, THUS RENDERING THE FUNCTION SINGLE-VALUED. THESE CUTS CAN BE LINES, CURVES ETC..

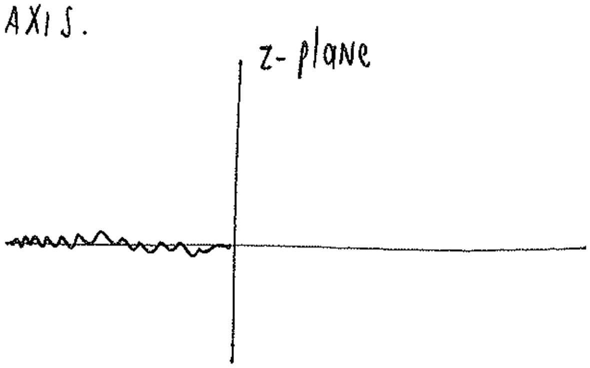
WE THEN CHOOSE A RANGE OF ARGUMENT TO UNAMBIGUOUSLY DEFINE THE FUNCTION AT EACH POINT IN THE CUT PLANE.



OR



CONSTRUCT A BRANCH OF $f = \log z$ THAT IS ANALYTIC EXCEPT ON THE NEGATIVE REAL AXIS AND IS REAL-VALUED ON POSITIVE REAL AXIS.



THIS IS

$$f(z) = \mathrm{LOG}\, z.$$

FOR WHEN $-\pi < \mathrm{ARG}\, z \leq \pi$ AND

SO FOR $z = x$ WITH $x > 0$ REAL,

$$\mathrm{IM}[f(z)] = 0.$$

<u>EXAMPLE</u> CONSIDER THE SINGLE-VALUED FUNCTION

$$f(z) = \mathrm{LOG}(1 - z^2)$$

WHERE IS THIS FUNCTION DISCONTINUOUS?

<u>SOLUTION</u> SINCE $\mathrm{LOG}(\zeta)$ IS ANALYTIC EXCEPT ON $\mathrm{IM}(\zeta) = 0$ AND $\mathrm{RE}(\zeta) \leq 0$, WE HAVE THAT $\mathrm{LOG}(1 - z^2)$ IS DISCONTINUOUS ONLY WHEN $\mathrm{IM}(1 - z^2) = 0$ AND $\mathrm{RE}(1 - z^2) < 0$.

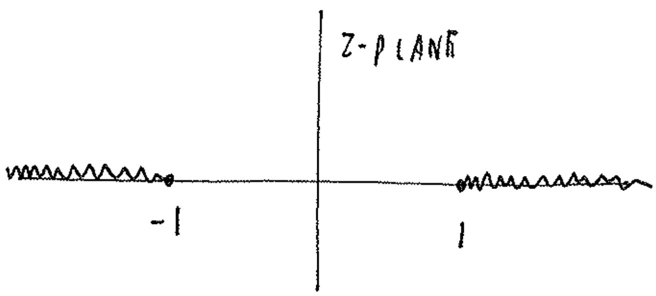
LET $z = x + iy$,

set $\text{IM}(1-z^2) = -2xy = 0$

$\text{RE}(1-z^2) = 1 - x^2 + y^2 < 0$

HENCE EITHER $x = 0$ OR $y = 0$. BUT IF $x = 0$ THEN $1 + y^2 < 0$ IS IMPOSSIBLE. HENCE $y = 0$ AND $1 - x^2 + y^2 = 1 - x^2 \leq 0$ IMPLIES $|x| \geq 1$.

THEREFORE THE BRANCH CUTS ARE AS SHOWN



$f(z) \equiv \text{LOG}(1 - z^2)$ IS ANALYTIC IN THE CUT PLANE AS SHOWN.

<u>EXAMPLE</u> LET $f(z) = \text{LOG}\left(\frac{z-1}{z-2}\right)$, WHERE LOG DENOTES THE PRINCIPAL BRANCH OF MULTI-VALUED log FUNCTION.

WHERE IS $f(z)$ ANALYTIC?

<u>SOLUTION</u> THE ONLY POSSIBLE PLACES WHERE f IS NOT ANALYTIC IS WHEN

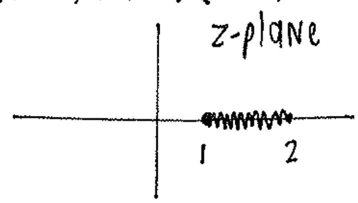
$$\text{IM}\left(\frac{z-1}{z-2}\right) = 0 \quad \text{AND} \quad \text{RE}\left(\frac{z-1}{z-2}\right) \leq 0.$$

HENCE $$\frac{(z-1)}{(z-2)}\frac{\overline{(z-2)}}{\overline{(z-2)}} = \frac{1}{|z-2|^2}\left[z\bar{z} - \bar{z} - 2z + 2\right] = \frac{1}{|z-2|}\left[x^2 + y^2 - (x - iy) - 2(x+iy) + 2\right]$$

SO $\text{IM}\left(\frac{z-1}{z-2}\right) = 0 \longrightarrow y = 0$

$\text{RE}\left(\frac{z-1}{z-2}\right) < 0$ WHEN $y = 0$ YIELDS $x^2 - 3x + 2 = (x-2)(x-1) \leq 0$.

THUS $f(z)$ IS NOT ANALYTIC ON BRANCH CUTS AS SHOWN

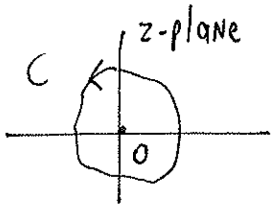


MULTI-VALUED FUNCTIONS

CONSIDER THE FUNCTION $w = f(z) = z^{1/2}$. (multi-valued)

THE POINT $z=0$ IS A <u>BRANCH POINT</u> SINCE IF WE ENCIRCLE $z=0$ BY A SIMPLE CLOSED CURVE C, THE CHANGE IN $z^{1/2}$, DENOTED BY $[z^{1/2}]\big|_C$ IS, FOR C COUNTER-CLOCKWISE

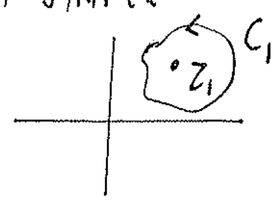
$$[z^{1/2}]\big|_C = \left[|z|e^{i\varphi/2}\right]_C = |z|e^{i\pi} \neq 0 \qquad (\varphi \mapsto \varphi + 2\pi)$$



FOR ANY OTHER POINT z_1, WE HAVE

$$[z^{1/2}]_{C_1} = 0$$

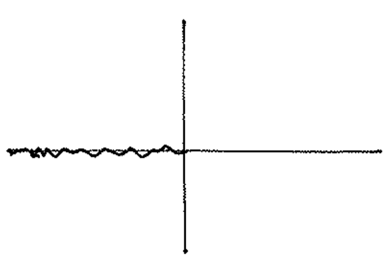
WHERE C_1 IS A SIMPLE CLOSED CURVE SURROUNDING z_1 AND NOT THE ORIGIN

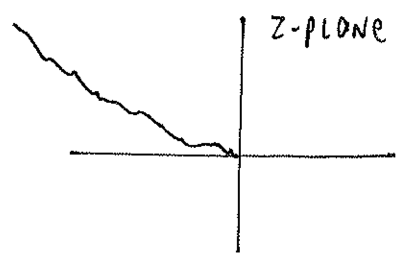


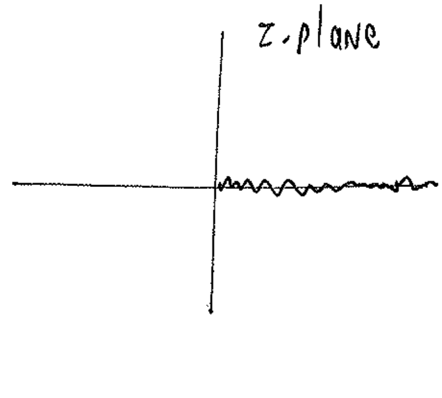
THUS $z_1 \neq 0$ IS NOT A BRANCH POINT.

WE MUST INTRODUCE A BRANCH-CUT EMANATING FROM $z=0$ AND EXTENDING TO ∞ TO PREVENT ENCIRCLING THE ORIGIN, AND HENCE RENDERING $z^{1/2}$ ANALYTIC IN THE CUT PLANE. THEN, WE CHOOSE A RANGE OF VALUES FOR THE ARGUMENT OF z TO MAKE IT UNIQUELY DEFINED IN CUT PLANE.

POSSIBLE BRANCH CUTS FOR $z^{1/2}$

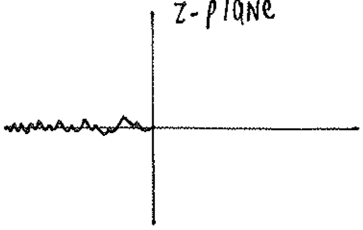






EXAMPLE CONSTRUCT A BRANCH OF $F(z)=z^{1/2}$ FOR WHICH $z^{1/2}$ IS ANALYTIC IN THE CUT PLANE $\mathbb{C}\setminus(-\infty,0)$ AND FOR WHICH $\mathrm{RE}(\sqrt{z})\geq 0$.

SOLUTION WE MUST HAVE THE CUT AS SHOWN



HENCE $z=|z|e^{i\varphi}$

AND EITHER $-\pi<\varphi\leq\pi$

OR $\pi<\varphi\leq 3\pi$.

WHICH RANGE OF ANGLES WORKS?

WE CALCULATE $z^{1/2}=|z|^{1/2}e^{i\varphi/2}$

THEN $\mathrm{RE}(z^{1/2})=|z|^{1/2}\cos(\varphi/2)$

HENCE, IF $-\pi<\varphi\leq\pi$, THEN $\cos(\varphi/2)\geq 0 \rightarrow \mathrm{RE}(\sqrt{z})\geq 0$.

WE THEN WRITE $(*)$ $z^{1/2}=|z|^{1/2}e^{i\varphi/2}$ WITH $-\pi<\varphi\leq\pi$.

REMARK (i) $(*)$ IS THE PRINCIPAL BRANCH OF $\sqrt{z}$.

IT COINCIDES PRECISELY WITH THE CHOICE OF BRANCH

$$z^{1/2}=e^{\frac{1}{2}\mathrm{LOG}(z)}=e^{\frac{1}{2}[\ln|z|+i\,\mathrm{ARG}(z)]} \qquad -\pi<\mathrm{ARG}\,z\leq\pi.$$

(ii) CALCULATE THE PRINCIPAL VALUE OF $(1+i)^{1/2}$.

SOLUTION $\mathrm{ARG}\,z=\pi/4$ $|z|=\sqrt{2}$. SO

$$(1+i)^{1/2}=(\sqrt{2})^{1/2}e^{i\pi/8}.$$

(iii) CONSTRUCT A BRANCH OF $z^{1/2}$ THAT IS ANALYTIC IN $\mathbb{C}\setminus(-\infty,0)$ BUT HAS $\mathrm{RE}(z^{1/2})<0$.

BY REPEATING ANALYSIS ABOVE, $z^{1/2}=|z|^{1/2}e^{i\varphi/2}$, $\pi<\varphi\leq 3\pi$

<u>EXAMPLE</u> SUPPOSE THAT $z^{1/2}$ DENOTES THE PRINCIPAL VALUE OF THE SQUARE ROOT. FIND ALL SOLUTIONS TO

$$(*) \qquad z^{1/2} + 2 - i = 0.$$

<u>SOLUTION</u> THE PRINCIPAL VALUE OF $z^{1/2}$ IS SUCH THAT IT IS ANALYTIC IN $\mathbb{C} \setminus (-\infty, 0)$ AND HAS $\mathrm{RE}(z^{1/2}) \geq 0$.

BY TAKING $\mathrm{RE}(\quad)$ OF BOTH SIDES IN $(*)$ WE OBTAIN THAT

$$\mathrm{RE}(z^{1/2}) + 2 = 0 \longrightarrow \mathrm{RE}(z^{1/2}) = -2 < 0 \longrightarrow \text{CONTRADICTION}.$$

THUS WITH THE PRINCIPAL VALUE OF $z^{1/2}$, $(*)$ HAS <u>NO SOLUTIONS</u>.

<u>REMARK</u> IT IS TEMPTING BUT <u>WRONG</u> TO CALCULATE AS

$$z^{1/2} = i - 2$$

$$\longrightarrow (z^{1/2})^2 = (i-2)^2 = 4 - 4i - 1 = 3 - 4i$$

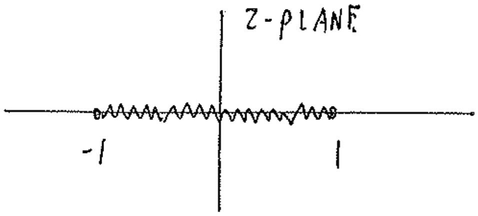
$$\text{SO} \quad z = 3 - 4i$$

<u>EXAMPLE</u> CONSTRUCT A BRANCH OF $f(z) = (z^2-1)^{1/2}$ THAT IS ANALYTIC IN $|z| > 1$ AND TAKES THE VALUE $f(2) = \sqrt{3}$.

<u>SOLUTION</u> WE USE $z_1^{1/2} z_2^{1/2} = (z_1 z_2)^{1/2}$ (multi-valued sets same) TO WRITE

$$f(z) = (z-1)^{1/2}(z+1)^{1/2}.$$

THE ONLY POINTS IN FINITE COMPLEX PLANE THAT ARE BRANCH POINTS ARE $z = -1$ AND $z = 1$. WE MUST HAVE NO BRANCH CUTS OUTSIDE $|z| > 1$, SO THE EASIEST CONSTRUCTION IS TO HAVE



METHOD 1 (RANGE OF ANGLES)

WE WRITE $(z^2-1)^{1/2} = (z-1)^{1/2}(z+1)^{1/2} = (r_1 e^{i\varphi_1})^{1/2}(r_2 e^{i\varphi_2})^{1/2}$.

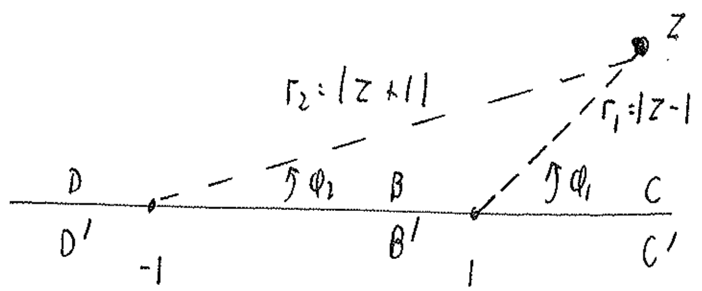
HENCE $$f(z) = (z^2-1)^{1/2} = (r_1 r_2)^{1/2} e^{i(\varphi_1+\varphi_2)/2} \qquad (*)$$

SPECIFYING A BRANCH IS EQUIVALENT TO CHOOSING A RANGE OF ANGLES.

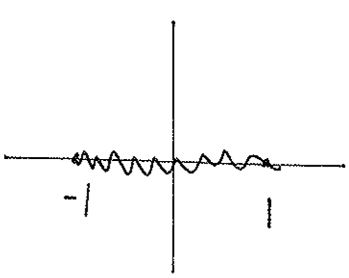
TRY $-\pi < \varphi_1 \le \pi, \qquad -\pi < \varphi_2 \le \pi$

- MUST CHECK THAT DISCONTINUITY IN f OCCURS BETWEEN $-1 \le X \le 1$
- MUST CHECK THAT $(*)$ GIVES $f(2) = \sqrt{3}$.

POINT	φ_1	φ_2	$e^{i(\varphi_1+\varphi_2)/2}$	
C	0	0	$e^{i0} = 1$	
C′	0	0	$e^{i0} = 1$	$\Rightarrow$ CONTINUOUS
B	π	0	$e^{i\pi/2} = i$	
B′	$-\pi$	0	$e^{-i\pi/2} = -i$	$\Rightarrow$ DISCONTINUOUS
D	π	π	$e^{i\pi} = -1$	
D′	$-\pi$	$-\pi$	$e^{-i\pi} = -1$	$\Rightarrow$ CONTINUOUS



THUS THE CHOICE $f(z) = (r_1 r_2)^{1/2} e^{i(\varphi_1+\varphi_2)/2}$ WITH $-\pi < \varphi_1 \le \pi$ AND $-\pi < \varphi_2 \le \pi$ HAS A BRANCH CUT FROM $-1 < X < 1$ AS DESIRED



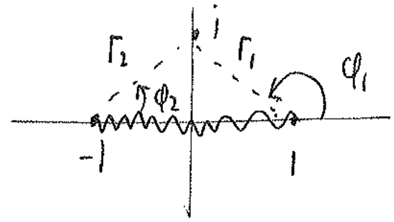
- NOW CALCULATE $f(2)$:

FOR $z = 2$ THEN $r_1 = |z-1| = 1$, $r_2 = |z+1| = 3$

AND $\varphi_1 = \varphi_2 = 0$ HENCE

$$f(2) = \sqrt{1\cdot 3}\, e^{i0} = \sqrt{3} \quad \text{AS DESIRED.}$$

• TO CALCULATE $f(i)$ WE DRAW



AND CALCULATE $r_1 = r_2 = \sqrt{2}$.

$$\varphi_1 = 3\pi/4, \quad \varphi_2 = \pi/4 \quad \text{SO} \quad f(i) = \left(\sqrt{2}\sqrt{2}\right)^{1/2} e^{i(3\pi/4+\pi/4)/2}$$

$$\longrightarrow f(i) = \sqrt{2}\, i.$$

METHOD 2 (CHOOSING A BRANCH OF $\log$)

THIS METHOD IS LESS INTUITIVE AS IT IS NOT CLEAR APRIORI WHICH BRANCH OF $\log$ TO TAKE.

FOR INSTANCE, CONSIDER SEVERAL POSSIBLE CHOICES:

(A) $\quad (z^2-1)^{1/2} = e^{\frac{1}{2}\log(z^2-1)} \longrightarrow f(z) = e^{1/2\,\mathrm{LOG}(z^2-1)}$

(B) $\quad (z^2-1)^{1/2} = [-(1-z^2)]^{1/2} \Longrightarrow f(z) = \pm i\, e^{1/2\,\mathrm{LOG}(1-z^2)}$

(C) $\quad (z^2-1)^{1/2} = [z^2(1-1/z^2)]^{1/2} \Rightarrow f(z) = \pm z\, e^{1/2\,\mathrm{LOG}(1-1/z^2)}$

(D) $\quad (z^2-1)^{1/2} = [-z^2(-1+1/z^2)]^{1/2} \Rightarrow f(z) = \pm iz\, e^{1/2\,\mathrm{LOG}(-1+1/z^2)}$

WHICH ONE WILL GIVE US $f(z)$ ANALYTIC IN $|z|>1$ WITH $f(2)=\sqrt{3}$?

THIS IS NOT CLEAR WITHOUT CONSIDERABLE EXTRA EFFORT.

• CONSIDER THE OBVIOUS CHOICE (A): $f(z) = e^{1/2\,\mathrm{LOG}(z^2-1)}$ TO SEE IF IT WORKS.

THEN $f(z)$ IS ANALYTIC EXCEPT WHEN

$$\mathrm{IM}(z^2-1) = 0 \quad \text{AND} \quad \mathrm{RE}(z^2-1) \leq 0.$$

LET $z = x + iy$

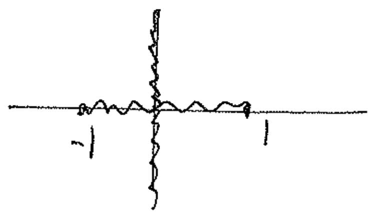
$$\text{IM}(z^2-1) = 0 \longrightarrow xy = 0 \longrightarrow \underline{\text{either}}\ x=0\ \underline{\text{OR}}\ y=0.$$

$$\text{RE}(z^2-1) = \text{RE}\left[x^2 - y^2 + 2ixy - 1\right] = x^2 - y^2 - 1 \leq 0.$$

$$\text{IF } x=0 \longrightarrow \text{RE}(z^2-1) = -y^2-1 \leq 0 \quad \text{FOR ALL } y$$

$$\text{IF } y=0 \longrightarrow \text{RE}(z^2-1) = x^2-1 \leq 0 \longrightarrow |x| \leq 1.$$

THUS, THIS CHOICE HAS BRANCH CUTS



WHICH IS __NOT__ WHAT WE WANT.

HENCE CHOICE A FAILS.

- __CONSIDER CHOICE (C)__ TRY $f(z) = \pm z e^{\frac{1}{2}\log(1 - 1/z^2)}$.

THIS IS ANALYTIC __EXCEPT__ WHEN $\text{IM}\left(1 - \frac{1}{z^2}\right) = 0$

$$\text{RE}\left(1 - \frac{1}{z^2}\right) \leq 0.$$

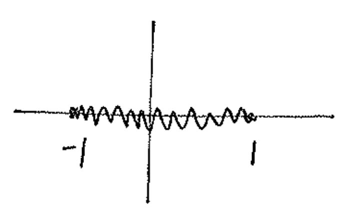
THUS $$\text{IM}\left(\frac{1}{z^2}\right) = \text{IM}\left(\frac{\bar{z}^2}{|z|^4}\right) = \text{IM}\left(\frac{(x - iy)^2}{|z|^4}\right) = 0 \longrightarrow xy = 0$$

$$\text{RE}\left(1 - \frac{1}{z^2}\right) = 1 - \text{RE}\left(\frac{(x-iy)^2}{|z|^4}\right) \leq 0.$$

- SET $y = 0 \longrightarrow \text{RE}\left(1 - \frac{1}{z^2}\right) \leq 0 \longrightarrow 1 - \frac{x^2}{x^4} \leq 0 \rightarrow |x| \leq 1.$
- SET $x = 0 \longrightarrow \text{RE}\left(1 - \frac{1}{z^2}\right) \leq 0 \rightarrow \text{RE}\left(1 - \frac{1}{(iy)^2}\right) = 1 + \frac{1}{y^2} \leq 0$

IMPOSSIBLE.

THUS $f(z) = \pm z e^{\frac{1}{2}\log(1 - 1/z^2)}$ HAS DESIRED BRANCH CUTS



NOW WE MUST TAKE $\pm$ SIGN CONSISTENT WITH $f(2) = \sqrt{3}$.

TRY $+$ SIGN $\longrightarrow$ $f(2) = 2 \quad e^{\frac{1}{2} \operatorname{LOG}(1 - 1/4)}$

$$= 2 \quad e^{\frac{1}{2}\left[\ln(3/4) + i \operatorname{ARG}(3/4)\right]}, \quad \operatorname{ARG}(3/4) = 0$$

$$= 2 \quad \left(e^{\ln(\sqrt{3/4})} \right) = 2 \quad \left(\sqrt{3/4} \right) = \sqrt{3} \ \checkmark.$$

THUS, THE DESIRED BRANCH IS

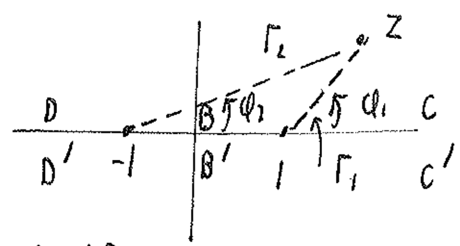
$$f(z) = z\, e^{\frac{1}{2} \operatorname{LOG}(1 - 1/z^2)}$$

THIS WAS NOT TERRIBLY CLEAR IN ADVANCE THAT THIS CHOICE WOULD WORK. NOTICE $f(i) = i\, e^{\frac{1}{2} \operatorname{LOG}(2)} = i\, e^{\frac{1}{2}(\ln 2)} = \sqrt{2}\, i$.

EXAMPLE CONSTRUCT A BRANCH OF $f(z) = (z^2 - 1)^{1/2}$ THAT IS ANALYTIC IN $|z| < 1$ AND THAT TAKES THE VALUE $f(0) = i$

SOLUTION

METHOD I (RANGE OF ANGLE METHOD)



WE WRITE $f(z) = (r_1 r_2)^{1/2} e^{i(\theta_1 + \theta_2)/2}$ $(*)$

WE TRY NOW THE RANGE $0 \le \theta_1 < 2\pi, \ -\pi < \theta_2 \le \pi$.

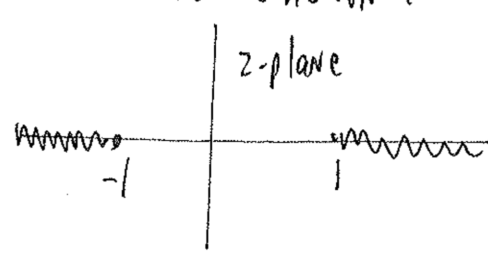
POINTS	θ_1	θ_2	$e^{i(\theta_1+\theta_2)/2}$
C	0	0	$e^{i0} = 1$
C'	2π	0	$e^{i\pi} = -1$
B	π	0	$e^{i\pi/2} = i$
B'	π	0	$e^{i\pi/2} = i$
D	π	π	$e^{i\pi} = -1$
D'	π	$-\pi$	$e^{i0} = 1$

$\not\ni$ DISCONTINUOUS (C, C')

$\ni$ CONTINUOUS (B, B')

$\not\ni$ DISCONTINUOUS (D, D')

$\Longrightarrow$ THIS YIELDS THE BRANCH CUT AS SHOWN (WHAT WE WANT)



HENCE, (K) IS ANALYTIC IN $|z|<1$.

NOW AT $z=0$ WE CALCULATE $\varphi_1=\pi,\ \varphi_2=0,\ r_1=r_2=1$ SO THAT

$$f(0)=(z^2-1)^{1/2}\Big|_{z=0}=(r_1r_2)^{1/2}e^{i(\varphi_1+\varphi_2)/2}=(1\cdot 1)^{1/2}e^{i\pi/2}=i, \text{ AS REQUIRED.}$$

<u>METHOD 2</u>

WE WRITE $(z^2-1)^{1/2}=[-(1-z^2)]^{1/2}=\pm i\,(1-z^2)^{1/2}$.

WE NOW CHOOSE THE PRINCIPAL VALUE $(1-z^2)^{1/2}=e^{\frac{1}{2}\mathrm{LOG}(1-z^2)}$.

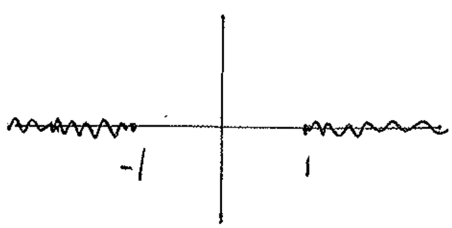
THEN, $(z^2-1)^{1/2}=(\pm i)\,e^{\frac{1}{2}\mathrm{LOG}(1-z^2)}$ THIS IS <u>CHOICE B</u> ON PAGE (M14.5)

CHOOSE $+i$ SINCE AT $z=0$ $\mathrm{LOG}(1)=\ln 1+i0=0$.

THEN $f(0)=i$ AS REQUIRED.

WE OBTAIN $(z^2-1)^{1/2}=i\,e^{\frac{1}{2}\mathrm{LOG}(1-z^2)}$.

BY THE EXAMPLE AT BOTTOM OF PAGE (M9) $\mathrm{LOG}(1-z^2)$ HAS BRANCH CUTS AS SHOWN



HENCE $(z^2-1)^{1/2}=i\,e^{\frac{1}{2}\mathrm{LOG}(1-z^2)}$ SATISFIES THE REQUIREMENTS.

- IN PARTICULAR IF $f(z)=(z^2-1)^{1/2}=i\,e^{\frac{1}{2}\mathrm{LOG}(1-z^2)}$

THEN $f(i)=i\,e^{\frac{1}{2}\mathrm{LOG}(1-i^2)}=i\,e^{\frac{1}{2}\mathrm{LOG}(2)}=i\,e^{\frac{1}{2}\ln 2}=\sqrt{2}\,i$.

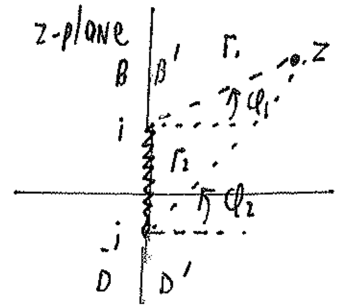
- RECALL $\mathrm{LOG}(1-z^2)$ IS ANALYTIC EXCEPT FOR POINTS z WHERE

$\mathrm{IM}(1-z^2)=0 \longrightarrow xy=0 \Longrightarrow$ EITHER $x=0$ OR $y=0$

$\mathrm{RE}(1-z^2)\le 0 \longrightarrow 1-(x^2-y^2)\le 0 \Longrightarrow$ IF $x=0 \longrightarrow 1+y^2\le 0$ IMPOSSIBLE

IF $y=0 \longrightarrow 1-x^2<0 \longrightarrow |x|\ge 1$.

EXAMPLE CONSTRUCT A BRANCH OF $f(z) = (z^2+1)^{1/2}$ THAT IS ANALYTIC IN $|z|>1$ AND FOR WHICH $f(2i) = \sqrt{3}\, i$



SOLUTION WE DO THIS QUICKLY.

METHOD 1 $f(z) = (z+i)^{1/2}(z-i)^{1/2} = (r_1 r_2)^{1/2} e^{i(\varphi_1+\varphi_2)/2}$

WE WANT A BRANCH CUT BETWEEN $-i$ AND i AS SHOWN.

IF WE CHOOSE $-\pi/2 < \varphi_1 \leq 3\pi/2$, $-\pi/2 < \varphi_2 \leq 3\pi/2$ THEN WE GET THE DESIRED BRANCH CUT. FOR THEN WE HAVE CONTINUITY AT $B|B'$ AND $D|D'$

FOR $z = 2i$ WE GET $\varphi_1 = \varphi_2 = \pi/2$, $r_1 = 1$, $r_2 = 3$. HENCE

$$f(2i) = (1\cdot 3)^{1/2} e^{i(\pi/2+\pi/2)/2} = i\sqrt{3}.$$

METHOD 2 THE CHOICE $(z^2+1)^{1/2} = e^{\frac{1}{2}\operatorname{Log}(z^2+1)}$ CLEARLY DOES NOT WORK SINCE $\operatorname{Log}(z^2+1)$ IS NOT ANALYTIC ON $z = iy$ WITH $|y|>1$.

INSTEAD WRITE $(z^2+1)^{1/2} = \left(z^2\left[1+\frac{1}{z^2}\right]\right)^{1/2} = z\left(1+1/z^2\right)^{1/2}$.

THEN CHOOSE $(z^2+1)^{1/2} = z\, e^{\frac{1}{2}\operatorname{Log}(1+1/z^2)}$

NOW $z\, e^{\frac{1}{2}\operatorname{Log}(1+1/z^2)}$ IS ANALYTIC EXCEPT ON THE SEGMENT FOR WHICH

$$\operatorname{Im}\left(1+\frac{1}{z^2}\right) = 0 \quad \text{AND} \quad \operatorname{Re}\left(1+\frac{1}{z^2}\right) \leq 0.$$

IF WE PUT $z = x+iy$ THEN $\operatorname{Im}\left(1+\frac{1}{z^2}\right) = \operatorname{Im}\left(\frac{\bar{z}^2}{|z|^4}\right) = \frac{1}{|z|^4}(-2xy) = 0$

HENCE EITHER $x = 0$ OR $y = 0$. BUT $\operatorname{Re}\left(1+\frac{1}{z^2}\right) = \operatorname{Re}\left(\frac{\bar{z}^2}{|z|^4}\right) + 1 = \frac{x^2-y^2}{(x^2+y^2)^2} + 1 < 0.$

CLEARLY $y = 0$ IS IMPOSSIBLE. SO $x = 0$ YIELDS $\frac{-y^2}{y^4} + 1 < 0 \rightarrow |y| < 1$.

WE CONCLUDE THAT $(z^2+1)^{1/2} = z\, e^{\frac{1}{2}\operatorname{Log}(1+1/z^2)}$ IS ANALYTIC IN $|z|>1$.

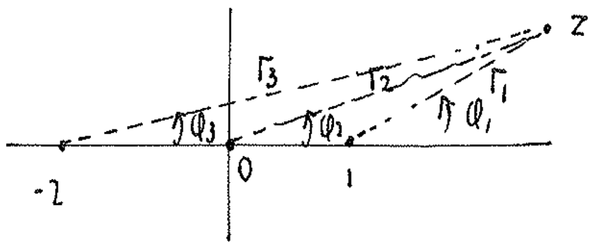
WE CALCULATE $f(2i) = 2i\, e^{\frac{1}{2}\operatorname{Log}(1-1/4)} = 2i\, e^{\frac{1}{2}\ln(3/4)} = 2i\sqrt{3}/2 = i\sqrt{3}$.

EXAMPLE CONSTRUCT A BRANCH OF $f(z) = (z^3 + z^2 - 2z)^{1/2}$

THAT HAS A BRANCH CUT FROM $(0,1)$ AND FROM $(-\infty,-2)$ ALONG

REAL AXIS AND FOR WHICH $f(2) = \sqrt{8}$.

SOLUTION $f(z) = \sqrt{z(z+2)(z-1)} = (r_1 r_2 r_3)^{1/2} e^{i(\varphi_1 + \varphi_2 + \varphi_3)/2}$



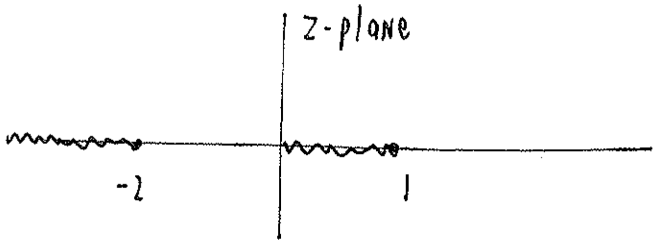
IF WE THEN CHOOSE

$-\pi < \varphi_1 \leq \pi$

$-\pi < \varphi_2 \leq \pi$

$-\pi < \varphi_3 \leq \pi$,

WE WILL OBTAIN THE BRANCH CUT STRUCTURE



TO CONVINCE YOURSELF DRAW A LITTLE TABLE OF VALUES.

WHEN $z = 2$ THEN $\varphi_1 = \varphi_2 = \varphi_3 = 0$, $r_1 = 1$, $r_2 = 2$, $r_3 = 4$.

HENCE $f(2) = (4 \cdot 2 \cdot 1)^{1/2} e^{i0} = \sqrt{8}$.

FINALLY, WE MAKE A FEW ADDITIONAL MISCELLANEOUS COMMENTS.

REMARK 1 NOT EVERYTHING WITH $\sqrt{z}$ HAS A BRANCH POINT AT $z = 0$.

FOR WHICH OF THE FOLLOWING IS $z = 0$ A BRANCH POINT?

(i) $f(z) = \sin(\sqrt{z})$ (ii) $f(z) = \sqrt{z}\,\sin(\sqrt{z})$

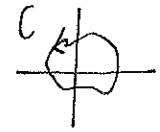
(iii) $f(z) = \cos(\sqrt{z})$.

IN (ii) WE USE SAME CHOICE OF BRANCH OF $\sqrt{z}$.

M19

SOLUTION ONLY $\sin(\sqrt{z})$ HAS A BRANCH POINT AT $z=0$.

LET'S CHECK. IN EACH CASE WE ENCIRCLE $z=0$ BY A SIMPLE CLOSED COUNTERCLOCKWISE CURVE AND WE CALCULATE $[f(z)]_C$ (THE CHANGE IN f AROUND THE CURVE).



(i) $$\left[\sin(\sqrt{z})\right]\Big|_C = \left[\sin\left(r^{1/2}e^{i\varphi/2}\right)\right]\Big|_C$$
$$= \sin\left(r^{1/2}e^{2\pi i/2}\right) - \sin\left(r^{1/2}e^{i0}\right)$$
$$= \sin\left(-r^{1/2}\right) - \sin\left(r^{1/2}\right) = -2\sin\left(r^{1/2}\right) \neq 0.$$

(ii) $$\left[\sqrt{z}\sin(\sqrt{z})\right]_C = \left[r^{1/2}e^{i\varphi/2}\sin\left(r^{1/2}e^{i\varphi/2}\right)\right]\Big|_C$$
$$= +r^{1/2}e^{2\pi i/2}\sin\left(r^{1/2}e^{2\pi i/2}\right) - r^{1/2}e^{i0}\sin\left(r^{1/2}e^{i0}\right)$$
$$= -r^{1/2}\sin\left(-r^{1/2}\right) - r^{1/2}\sin\left(r^{1/2}\right) = 0.$$

(iii) $$\left[\cos(\sqrt{z})\right]\Big|_C = \left[\cos\left(r^{1/2}e^{i\varphi/2}\right)\right]\Big|_C$$
$$= \cos\left(r^{1/2}e^{2\pi i/2}\right) - \cos\left(r^{1/2}\right) = \cos\left(-r^{1/2}\right) - \cos\left(r^{1/2}\right)$$
$$= 0 \quad \text{SINCE} \quad \cos(w) = \cos(-w).$$

REMARK 2 TO CLASSIFY WHETHER $z=\infty$ IS A BRANCH POINT OF $f(z)$ WE MUST TAKE A VERY LARGE CIRCLE $|z|=R$ $R>>1$ AND SEE IF $f(z)$ RETURNS TO ITS ORIGINAL VALUE AS WE TRAVERSE THE CIRCLE.

EQUIVALENTLY, $z=\infty$ IS A BRANCH POINT OF $f(z)$ IFF $\zeta=0$ IS A BRANCH POINT OF $f(1/\zeta)$ (I.E. LET $z=1/\zeta$).

EXAMPLE IS $z=\infty$ A BRANCH POINT FOR

(i) $f(z)=\sqrt{(z+1)(z+2)(z-3)}$

(ii) $f(z)=\log\left(\dfrac{z+1}{z-1}\right)$

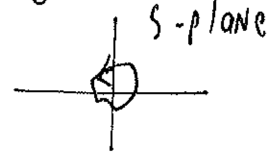
(iii) $f(z)=(z^3-z)^{1/3}$

SOLUTION

(i) LET $\zeta=1/z$ SO $f(1/\zeta)=\sqrt{(1+1/\zeta)(2+1/\zeta)(-3+1/\zeta)}=\zeta^{-3/2}\sqrt{(1+\zeta)(1+2\zeta)(1-3\zeta)}$

FOR $|\zeta|\ll 1$, $f(1/\zeta)\cong\zeta^{-3/2}$ SO THAT $[f(1/\zeta)]\big|_C\neq 0$ WHERE

C IS THE SMALL CIRCLE $|\zeta|=\delta$ $\delta\ll 1$.



SO $z=\infty$ is A BP FOR $f(z)$

(ii) $f(z)=\log(1+z)-\log(z-1)$

LET $z=1/\zeta$, THEN
$$f(1/\zeta)=\log(1+1/\zeta)-\log(1/\zeta-1)$$
$$=\log\left(\frac{1+\zeta}{\zeta}\right)-\log\left(\frac{1-\zeta}{\zeta}\right)$$
$$=\log(1+\zeta)-\log(1-\zeta).$$

LET $C:|\zeta|=\delta$, WITH $\delta\ll 1$. THEN $[f(1/\zeta)]_C=0$.

$\zeta=0$ IS NOT A BP OF $f(1/\zeta)\longrightarrow z=\infty$ IS NOT A BP OF $f(z)$.

(iii) LET $z=1/\zeta$, $f(1/\zeta)=\left(1/\zeta^3-1/\zeta\right)^{1/3}=\dfrac{(1-\zeta^2)^{1/3}}{(\zeta^3)^{1/3}}=\dfrac{(1-\zeta^2)^{1/3}}{\zeta}$.

NOW LET $C:|\zeta|=\delta$ WITH $\delta\ll 1$,

THEN $[f(1/\zeta)]\big|_C=0$. HENCE $\zeta=0$ IS NOT A BP OF $f(1/\zeta)$

$\longrightarrow z=\infty$ IS NOT A BP OF $f(z)$.

<u>EXAMPLE</u> FIND ALL POSSIBLE VALUES OF

(i) $\cos w = 2i$. (ii) $\sin w = i$

<u>SOLUTION</u>

(i) LET $z = \cos w = \dfrac{e^{iw} + e^{-iw}}{2}$ SO $\dfrac{e^{iw} + e^{-iw}}{2} = 2i$.

HENCE $e^{iw} + e^{-iw} = 4i \longrightarrow e^{2iw} - 4ie^{iw} + 1 = 0$.

LET $\Lambda = e^{iw} \longrightarrow \Lambda^2 - 4i\Lambda + 1 = 0$.

THUS $\Lambda = \dfrac{4i \pm \sqrt{-16 - 4}}{2} = 2i \pm i\sqrt{5}$

NOW $e^{iw} = (2 \pm \sqrt{5})i$

<u>+ SIGN</u> $e^{iw} = (2 + \sqrt{5})i$

$iw = \log((2+\sqrt{5})i) = \ln(2+\sqrt{5}) + i(\pi/2 + 2K\pi)$ $K = 0, \pm 1, \pm 2, \ldots$

THUS $w = -i\ln(2+\sqrt{5}) + \pi/2 + 2K\pi$, $K = 0, \pm 1, \pm 2 \ldots$

<u>- SIGN</u> $e^{iw} = (2 - \sqrt{5})i \longrightarrow iw = \log((2-\sqrt{5})i) = \ln(\sqrt{5}-2) + i\left(-\frac{\pi}{2} + 2K\pi\right)$

THUS $w = -i\ln(\sqrt{5}-2) - \dfrac{\pi}{2} + 2K\pi$ $K = 0, \pm 1, \pm 2, \ldots$

SINCE $\ln(\sqrt{5}-2) = \ln\left((\sqrt{5}-2)(\sqrt{5}+2)/(\sqrt{5}+2)\right) = +\ln\left(1/\sqrt{5}+2\right) = -\ln(\sqrt{5}+2)$

THEN WE CAN WRITE +SIGN AND -SIGN TOGETHER AS

(*) $w = \pm i\ln(\sqrt{5}+2) \pm \dfrac{\pi}{2} + 2K\pi$ $K = 0, \pm 1, \ldots$

THE SYMMETRY IN (*) FOLLOWS FROM IDENTITY THAT

$\cos w_0 = \cos(-w_0)$.

M22

(ii) FOR THE $\sin w = i$ WE PUT

$$\frac{e^{iw} - e^{-iw}}{2i} = i \longrightarrow e^{iw} - e^{-iw} = -2.$$

THUS $$e^{2iw} + 2e^{iw} - 1 = 0 \rightarrow \Lambda^2 + 2\Lambda - 1 = 0 \quad \text{WITH } \Lambda = e^{iw}.$$

SO $$\Lambda = \frac{-2 \pm \sqrt{4+4}}{2} = -1 \pm \sqrt{2}.$$

+ SIGN $$e^{iw} = -1 + \sqrt{2} \longrightarrow iw = \log(\sqrt{2} - 1) = \ln(\sqrt{2} - 1) + 2k\pi i.$$

SO $$w = -i \ln(\sqrt{2} - 1) + 2k\pi, \quad k = 0, \pm 1, \pm 2, \ldots$$

− SIGN $$e^{iw} = -1 - \sqrt{2} \longrightarrow iw = \log(-1 - \sqrt{2}) = \ln(\sqrt{2} + 1) + i(-\pi + 2k\pi)$$

SO $$w = -i \ln(\sqrt{2} + 1) + [-\pi + 2k\pi] \quad k = 0, \pm 1, \pm 2, \ldots$$

SINCE $\ln(\sqrt{2} + 1) = -\ln(\sqrt{2} - 1)$ IT IS CLEARLY THAT THE SYMMETRY IN THE TWO RESULTS FOLLOWS FROM THE FACT THAT IF $w = w_0$ IS A ROOT OF $\sin(w) = z$ THEN SO IS $w = \pi - w_0$