University of British Columbia Math 300 Final Exam

December 16, 2019, 7:00–9:30pm ANGU 098 (2.5 hours)

Last Name (print):

First Name (print):

Student ID Number: **Signature:**

Rules governing examinations

- Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
- Candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no candidate shall be permitted to enter the examination room once the examination has begun.
- Candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
- Candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
  (a) speaking or communicating with other candidates, unless otherwise authorized;
  (b) purposely exposing written papers to the view of other candidates or imaging devices;
  (c) purposely viewing the written papers of other candidates;
  (d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
  (e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)–(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
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- Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
- Candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

Question	Points	Score
1	5	
2	10	
3	5	
4	10	
5	10	
6	10	
7	20	
8	10	
9	10	
10	10	
Total:	100	

Additional Instructions:

- No notes, books or calculators are allowed.
- Read the questions carefully and make sure you provide all the information that is asked for in the question.
- Show all your work. Correct answers without explanation or accompanying work could receive no credit.
- Answer the questions in the space provided. Continue on the back of the page if necessary.

1. (5 points) Let $[1, i]$ be a straight line segment joining 1 to i in the complex plane. Compute the following integrals.

   (a) $\displaystyle\int_{[1,i]} z \, dz,$

   Solution: The analytic function $f(z) = z$ has antiderivative $z^2/2$. Thus

   $$\int_{[1,i]} z \, dz = z^2/2\Big|_{z=1}^{i} = (i^2)/2 - 1/2 = -1$$

   (b) $\displaystyle\int_{[1,i]} \overline{z} \, dz,$

   Solution: This integral is not path independent so we must choose a parametrization. We can use $z(t) = 1 + t(i - 1)$ for $t \in [0, 1]$. then $\overline{z}(t) = 1 + t(-i - 1)$ and $z'(t) = i - 1$. Thus

   $$\int_{[1,i]} \overline{z} \, dz \int_0^1 \overline{z}(t) z'(t) dt = i.$$

   (c) $\displaystyle\int_{[1,i]} |z|^2 \, dz.$

   Solution: Using the same parametrization as in the previous part, we get $|z(t)|^2 = t^2 + (1 - t)^2$ so

   $$\int_{[1,i]} |z|^2 \, dz = \int_0^1 (t^2 + (1 - t)^2)(i - 1) dt = (2/3)(i - 1)$$

2. (a) (5 points) The generalized Cauchy integral formula has the form

$$f^{(n)}(z) = C_n \int_\Gamma \frac{f(w)}{(w-z)^{n+1}} dw.$$

where C_n is a constant and Γ is a suitable contour. What conditions on $f(z)$, Γ and C_n make this true?

Solution:

- Γ is a simple closed curve.
- Γ is traversed once with positive (counterclockwise) orientation.
- The point z lies inside Γ
- The function $f(z)$ is analytic on and inside Γ
- $C_n = \dfrac{n!}{2\pi i}$.

(b) (5 points) Show that an entire function $f(z)$ that satisfies the bound $|f(z)| \le |z|^2$ when $|z| > 1$ must be a polynomial of degree at most 2. (Hint: show that $f^{(3)}(z) = 0$ for every z.)

Solution: Let Γ_R be the circle $\{w : |w - z| = R\}$, traversed once counterclockwise. Then by the generalized Cauchy Integral Formula we find

$$f^{(3)}(z) = \frac{3!}{2\pi i} \int_{\Gamma_R} \frac{f(w)}{(w-z)^4} dw.$$

When $w \in \Gamma_R$ we have the lower bound

$$|w| = |w - z + z| \ge |w - z| - |z| = R - |z|$$

as well as the upper bound

$$|w| = |w - z + z| \le |w - z| + |z| = R + |z|.$$

The lower bound implies that for sufficiently large R all $w \in \Gamma_R$ satisfy $|w| > 1$ so that $|f(w)| \le |w|^2$. Now the upper bound gives that $|f(w)| \le (R + |z|)^2$ so we can use our basic estimate:

$$\begin{aligned} \left|f^{(3)}(z)\right| &= \left| \frac{3!}{2\pi i} \int_{\Gamma_R} \frac{f(w)}{(w-z)^4} dw \right| \\ &\le \frac{6}{2\pi} \frac{(R + |z|)^2}{R^4} \times \text{length}(\Gamma_R) \le 6(R + |z|)^2 R / R^4 \to 0 \end{aligned}$$

as $R \to \infty$. Thus $f^{(3)}(z) = 0$ for all z. This implies that $f(z)$ is a polynomial of degree at most 2.

To see this we note that $f^{(3)}(z) = 0$ implies that $f^{(2)}(z)$ is a constant, say a. Then $f' - az$ satisfies $(f' - az)' = f^{(2)}(z) - a = 0$. Thus $f' - az$ is also a constant, say b. But then $(f - az^2/2 - bz)' = f' - az - b = 0$. So $(f - az^2/2 - bz)$ is yet another constant, say c. This gives $f = az^2/2 + bz + c$. In other words, f is a polynomial of degree at most 2.

3. (5 points) Give an example of an open set U and an function $f(z)$ analytic on U with no antiderivative (i.e., there is no function $F(z)$ with $F'(z) = f(z)$ for $z \in U$). Can you find such an $f(z)$ if U is a disk?

Solution: An example is $f(z) = 1/z$ and $U = \mathbb{C} \backslash \{0\}$. Then $f(z)$ is analytic on U but doesn't have an antiderivative, because having an antiderivative is equivalent to all integrals about closed loops being zero. In this case the integral of $1/z$ about the unit circle is $2\pi i$ so f has no antiderivative

If U is a disk, then the integral of any analytic f around a closed loop is zero. Thus the antiderivative exisits. Another way to see this is to note that $f(z)$ has a uniformly convergent power series in the disk which can be integrated term by term to obtain an antiderivative.

4. (10 points) Suppose that $p(z)$ and $q(z)$ are polynomials with $\deg(q) > \deg(p)$. In addition, assume that all the zeros of q lie inside the unit circle. Show that

$$\oint_{|z|=1} \frac{p(z)}{q(z)(z-2i)} = -2\pi i \frac{p(2i)}{q(2i)}.$$

where the contour is traversed once counterclockwise.

Solution: Since all the zeros of q are inside the unit circle, the only singularity between $|z| = 1$ and $|z| = R$ is at $2i$, provided $R > 2$. So for $R > 2$

$$\oint_{|z|=R} \frac{p(z)}{q(z)(z-2i)} = \oint_{|z|=1} \frac{p(z)}{q(z)(z-2i)} + 2\pi i \operatorname{Res}\left[\frac{p(z)}{q(z)(z-2i)}, z = 2i\right].$$

Since $\deg(q(z)(z-2i)) \geq \deg(p) + 2$ a simple estimate shows $\oint_{|z|=R} \frac{p(z)}{q(z)(z-2i)} \to 0$ as $R \to \infty$. Thus the right side of the equation must be zero. This gives the desired equality, since $q(2i) \neq 0$ implies that

$$\operatorname{Res}\left[\frac{p(z)}{q(z)(z-2i)}, z = 2i\right] = \frac{p(2i)}{q(2i)}.$$

5. For values of z near 3 we can write the following convergent Laurent series:

$$\frac{2z-4+2i}{(z+2i)(z-4)} = \sum_{n=-\infty}^{\infty} a_n z^n = \sum_{n=-\infty}^{\infty} b_n (z-3)^n = \sum_{n=-\infty}^{\infty} c_n (z-4)^n$$

(a) (3 points) Determine the region of convergence for each series.

Solution: The region of convergence is the largest annulus A such that

- $\dfrac{2z-4+2i}{(z+2i)(z-4)}$ is analytic on A
- A contains points near three,
- for a series with powers $(z-z_0)^n$, the boundaries of A are circles centred at z_0.

So
$\sum_{n=-\infty}^{\infty} a_n z^n$ converges for $2 < |z| < 4$,
$\sum_{n=-\infty}^{\infty} b_n (z-3)^n$ converges for $|z-3| < 1$ and
$\sum_{n=-\infty}^{\infty} c_n (z-4)^n$ converges for $0 < |z-4| < \sqrt{20}$.

(b) (2 points) Which series is actually a Taylor series?

Solution: The series $\sum_{n=-\infty}^{\infty} b_n (z-3)^n$ is a Taylor series because the function we are expanding is analytic at $z = 3$.

(c) (5 points) Find the coefficients a_n.

Solution: We start with a partial fraction decomposition:

$$\frac{2z-4+2i}{(z+2i)(z-4)} = \frac{1}{z-4} + \frac{1}{z+2i}$$

Then, as long as $|z| < 4$,

$$\frac{1}{z-4} = \frac{-1}{4}\frac{1}{1-z/4} = \sum_{k=0}^{\infty} -4^{-k-1} z^k$$

while, provided $|z| > 2$,

$$\frac{1}{z+2i} = \left(\frac{1}{z}\right)\left(\frac{1}{1+2i/z}\right) = \left(\frac{1}{z}\right)\sum_{k=0}^{\infty}(2i/z)^k = \sum_{k=0}^{\infty}(2i)^k/z^{k+1}.$$

This leads to

$$a_n = \begin{cases} -4^{-n-1} & n \geq 0 \\ (-2i)^{-n-1} & n \leq -1 \end{cases}$$

in the region where $2 < |z| < 4$.

6. (a) (5 points) Find all solutions z to the equation

$$z^3 = 27i.$$

Solution: Write $z = re^{i\theta}$. Then $z^3 = 27i$ if and only if $r^3e^{3i\theta} = 27e^{i\pi/2}$. This happens when $r = 3$ and $3\theta = \pi/2 + 2\pi k$ for some $k \in \mathbb{Z}$. So the solutions are $3e^{i\pi/6+i2\pi k/3}$, $k \in \mathbb{Z}$. Distinct solutions are given by $k = 0, 1, 2$.

(b) (5 points) Which of the solutions in the previous part is the principal value of $(27i)^{1/3}$? Explain.

Solution: The principal value of $\log(27i)$ is $\mathrm{Log}(27i) = 3\ln(3) + i\pi/2$. So the principal value of $(27i)^{1/3}$ is

$$(27i)^{1/3} = e^{(1/3)\,\mathrm{Log}(27i)} = e^{(1/3)(3\ln(3)+i\pi/2)} = e^{\ln(3)}e^{i\pi/6} = 3e^{i\pi/6}.$$

7. (a) (5 points) Find the first three non-zero terms in the Taylor expansion of $\frac{z\cos(z)}{\sin(z)}$ about $z = 0$. What is the radius of convergence of this series?

Solution: The radius of convergence is π since this is the distance from $z = 0$ to the nearest singularity.

There are a few ways to get the series low order. If we recognize the series for $\cos(z)$ and $\sin(z)/z$ both involve only even powers, then the quotient will also have only even powers. So the desired series satisfies

$$\begin{aligned} &1 - \frac{z^2}{2} + \frac{z^4}{4!} + O(z^6) \\ &\qquad = \left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} + O(z^6)\right)\left(a_0 + a_2 z^2 + a_4 z^4 + O(z^6)\right) \\ &\qquad = a_0 + \left(-\frac{a_0}{3!} + a_2\right) z^2 + \left(\frac{a_0}{5!} + \frac{-a_2}{3!} a_2 + a_4\right) z^4 \end{aligned}$$

Comparing coefficients yields $a_0 = 1$, $a_2 = -1/3$ and $a_4 = -1/45$.

(b) (5 points) Let

$$f(z) = \frac{\cos(z)}{z^4 \sin(z)}.$$

Find all the singularities of $f(z)$, classify them (as removable, pole of order n or essential) and compute the residue at each one.

Solution: The singularities are at the zeros of the denominatior which occur at $z \in \pi\mathbb{Z}$ (i.e., at integer multiples of π.) The point $z = 0$ is different, though, since it is also a zero of z^4.

Expanding at $z = 0$ and using the results of the last part gives

$$f(z) = \frac{1}{z^5} \frac{z \cos(z)}{\sin(z)} = \frac{1}{z^5} - \frac{1}{3} \frac{1}{z^3} - \frac{1}{45} \frac{1}{z} + O(1)$$

So $z = 0$ is a pole of order 5 with residue $-1/45$.

The other singularities $z = k \in \pi\mathbb{Z}\backslash\{0\}$ are simple poles with residues

$$\frac{\cos(\pi k)}{4(\pi k)^3 \sin(\pi k) + (\pi k)^4 \cos(\pi k)} = \frac{1}{(\pi k)^4}$$

(c) (5 points) Let Γ_N for $N \in \mathbb{N}$ denote the square in the complex plane with corners $\pi(N+1/2)(\pm 1 \pm i)$, traversed once counterclockwise. Calculate the integral

$$\oint_{\Gamma_N} f(z)dz.$$

Solution: By the residue formula this is equal to $2\pi i$ times the sum of the residues at the singularities inside Γ_1. Thus

$$\oint_{\Gamma_N} f(z)dz = 2\pi i\Big(\sum_{k=-N}^{-1} \frac{1}{k^4\pi^4} - \frac{1}{45} + \sum_{k=0}^{N} \frac{1}{k^4\pi^4}\Big) = 2\pi i\Big(\frac{2}{\pi^4}\sum_{k=0}^{N} \frac{1}{k^4} - \frac{1}{45}\Big)$$

(d) (5 points) Given the estimate

$$\max_{z\in\Gamma_N} \left|\frac{\cos(z)}{\sin(z)}\right| \le C,$$

where C does not depend on N, show how you can use the calculation in the previous part to evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$.

Solution: Using the facts that $|z| \ge \pi N$ when $z \in \Gamma_N$, and that $\text{length}(\Gamma_N) = C_1 N$, our basic estimate yields

$$\begin{aligned}\left|\oint_{\Gamma_N} f(z)dz\right| &\le \max_{z\in\Gamma_n} |f(z)| \times \text{ length}(\Gamma_N) \\ &\le \max_{z\in\Gamma_n} \left|\frac{\cos(z)}{z^4 \sin(z)}\right| \times C_1 N \\ &\le \frac{CC_1N}{(\pi N)^4} \to 0\end{aligned}$$

as $N \to \infty$. This implies $\lim_{N\to\infty} \frac{2}{\pi^4}\sum_{k=0}^{N} \frac{1}{k^4} - \frac{1}{45} = 0$ or

$$\sum_{k=0}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}.$$

8. (10 points) Compute the integral. Show all your steps and indicate what estimates are needed. You need not prove the required estimates.

$$I = \text{p.v.} \int_{-\infty}^{\infty} \frac{\cos(2x)}{x - 3i} dx.$$

Solution: Let C_R^+ and C_R^- denote semi-circle contours, centred at 0 in the upper and lower half planes and oriented in the counterclockwise direction

$$\begin{aligned} I &= \lim_{R\to\infty} \frac{1}{2} \left[\int_{[-R,R]} \frac{e^{2iz}}{z-3i} dz + \int_{[-R,R]} \frac{e^{-2iz}}{z-3i} dz \right] \\ &= \lim_{R\to\infty} \frac{1}{2} \left[\int_{[-R,R]+C_R^+} \frac{e^{2iz}}{z-3i} dz + \int_{[-R,R]-C_R^-} \frac{e^{-2iz}}{z-3i} dz \right] \end{aligned}$$

since $\lim_{R\to\infty} \int_{C_R^+} \frac{e^{2iz}}{z-3i} dz = \lim_{R\to\infty} \int_{C_R^-} \frac{e^{-2iz}}{z-3i} dz$ by Jordan's lemma. Since the only singularity of the integrand is in the upper half plane, the second integral above is zero by Cauchy's theorem. Finally the residue formula yields

$$I = \frac{1}{2} 2\pi i \,\text{Res} \left[\frac{e^{2iz}}{z-3i}, 3i \right] = \pi i e^{-6}$$

9. (10 points) Compute the integral. Show all your steps and indicate what estimates are needed. You need not prove the required estimates

$$I = \int_0^\infty \frac{1-\cos(x)}{x^2} dx$$

Solution: Since the integrand is even,

$$I = \frac{1}{2} \lim_{R\to\infty} \int_{[-R,R]} \frac{1-\cos(x)}{x^2} dx$$

We want to write $(1-\cos(x))/x^2$ as $\mathrm{Re}(1-e^{ix})/x^2$ and then exchange Re and $\int$. The problem is that although $\frac{1-\cos(z)}{z^2}$ is analytic at 0, the function $\frac{1-e^{iz}}{z^2}$ has a (simple) pole. So to use this procedure, we must avoid the point 0. This can be done by opening a small gap in the contour and closing it again. Let C_a^+ denote the semicircle of radius a in the upper half plane oriented counterclockwise.

$$\begin{aligned}
I &= \frac{1}{2} \lim_{R\to\infty,\epsilon\to 0} \int_{[-R,-\epsilon]+[\epsilon,R]} \frac{1-\cos(x)}{x^2} dx \\
&= \frac{1}{2} \lim_{R\to\infty,\epsilon\to 0} \int_{[-R,-\epsilon]+[\epsilon,R]} \frac{1-\cos(x)}{x^2} dx \\
&= \frac{1}{2} \lim_{R\to\infty,\epsilon\to 0} \mathrm{Re}\left[\int_{[-R,-\epsilon]+[\epsilon,R]} \frac{1-e^{ix}}{x^2} dx\right] \\
&= \frac{1}{2} \lim_{R\to\infty,\epsilon\to 0} \mathrm{Re}\left[\int_{[-R,-\epsilon]-C_\epsilon^+ +[\epsilon,R]+C_R^+} \frac{1-e^{iz}}{z^2} dz \right. \\
&\qquad \left. + \int_{C_\epsilon^+} \frac{1-e^{iz}}{z^2} dz - \int_{C_R^+} \frac{1-e^{iz}}{z^2} dz\right] \\
&= \frac{1}{2} = \frac{1}{2} 0 + i\pi\,\mathrm{Res}[\frac{1-e^{iz}}{z^2}, 0] - 0 \\
&= \frac{\pi}{2}
\end{aligned}$$

Here we used Cauchy's theorem for the first integral, the indented contour formula for the second and Jordan's lemma for the third

10. (10 points) Compute the integral

$$I = \int_0^{2\pi} \frac{1}{2+\sin(\theta)} d\theta$$

by rewriting it as a contour integral.

Solution:

$$\begin{aligned} I &= \oint_{|z|=1} \frac{1}{2+(z-1/z)/2i} \frac{dz}{iz} \\ &= \oint_{|z|=1} \frac{2}{z^2+4iz-1} dz \\ &= \oint_{|z|=1} \frac{2}{(z-r_1)(z-r_2)} dz \end{aligned}$$

where $r_1 = -2i + \sqrt{3}i$, $r_2 = -2i - \sqrt{3}i$. Since r_1 is inside the unit circle and r_2 isn't, the residue formula yields

$$I = 2\pi i \frac{2}{r1 - r2} = 2\pi i \frac{2}{2i\sqrt{3}} = \frac{2\pi}{\sqrt{3}}$$