Residue Examples

filename: `residueexamples.tex`

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All contours are traversed in the counterclockwise (positive) direction.

1.

$$I = \oint_{|z|=1} ze^{1/z} dz$$

Essential singularity at $z = 0$. To compute residue, expand

$$ze^{1/z} = z\left(1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \cdots\right) = z + 1 + \frac{1}{2!}\frac{1}{z} + \cdots .$$

This gives $\text{Res}\left[ze^{1/z}; 0\right] = 1/2$ so $I = 2\pi i/2 = \pi i$.

2.

$$I = \oint_{|z|=2} \frac{3z-1}{z(z-1)^3} dz$$

Singularities at $z = 0$ and $z = 1$. Could evaluate by finding the residues at each singularity. But since both singularities are inside the contour, we also have

$$I = 2\pi i \,\text{Res}\left[\frac{3z-1}{z(z-1)^3}; \infty\right].$$

To compute the residue at infinity check that

$$\lim_{|z|\to\infty} \frac{3z-1}{z(z-1)^3} = 0.$$

This implies

$$\lim_{|z|\to\infty} \text{Res}\left[\frac{3z-1}{z(z-1)^3}; \infty\right] = \lim_{z\to\infty} z\frac{3z-1}{z(z-1)^3} = 0.$$

So $I = 0$.

3.

$$I = \oint_{|z|=2} \frac{5z-2}{z(z-1)} dz$$

Simple poles at $z = 0$ and $z = 1$. Residues are

$$\text{Res}\left[\frac{5z-2}{z(z-1)}; 0\right] = \frac{-2}{-1} = 2$$

and

$$\text{Res}\left[\frac{5z-2}{z(z-1)}; 1\right] = \frac{3}{1} = 3.$$

So

$$I = 2\pi i(2+3) = 10\pi i.$$

Alternatively, since both singularities are inside the contour, and

$$\lim_{|z|\to\infty} \frac{5z-2}{z(z-1)} = 0,$$

we have

$$I = 2\pi i \,\text{Res}\left[\frac{5z-2}{z(z-1)}; \infty\right] = 2\pi i \lim_{z\to\infty} z\frac{5z-2}{z(z-1)} = 10\pi i$$

4.

$$I = \oint_{|z|=1} \frac{1}{z^2 \sin(z)} dz$$

Pole of order 3 at $z = 0$. Compute the series at $z = 0$ directly:

$$\begin{aligned}\frac{1}{z^2\sin(z)} &= \frac{1}{z^2(z - z^3/3! + z^5/5! \ldots)} = \frac{1}{z^3(1 - z^2/3! + z^4/5! \ldots)} \\ &= \frac{1}{z^3}(1 + (z^2/3! - z^4/5! \ldots) + (z^2/3! - z^4/5! \ldots)^2 + \cdots) \\ &= 1/z^3 + 1/(6z) + \text{analytic.}\end{aligned}$$

So $\text{Res}\left[\frac{1}{z^2\sin(z)}; 0\right] = 1/6$ and $I = i\pi/3$.

5.

$$I = \oint_{|z|=3} \frac{1}{z^2+z+1} dz$$

Simple poles at $z = 1/2 \pm \sqrt{3}/2$. Both poles are on the unit circle, inside the contour. Check that

$$\lim_{z\to\infty} \frac{1}{z^2+z+1} = 0.$$

Thus

$$I = 2\pi i \operatorname{Res}\left[\frac{1}{z^2+z+1}; \infty\right] = 2\pi i \lim_{z\to\infty} z \frac{1}{z^2+z+1} = 0.$$

6.

$$I = \oint_{|z|=1} e^{1/z} \sin(1/z) dz$$

Essential singularity at $z = 0$. To find the residue we can compute the series using a Cauchy product

$$\begin{aligned} e^{1/z}\sin(1/z) &= \Big(1 + 1/z + 1/(2z^2) + \cdots\Big)\Big(1/z - 1/(3!z^3) + \cdots\Big) \\ &= 1/z + \text{lower order terms} \end{aligned}$$

Thus $\operatorname{Res}[e^{1/z}\sin(1/z), 0] = 1$ and $I = 2\pi i$.

7.

$$I = \oint_{|z|=1} \frac{z-1}{\sin(z)} dz$$

Simple poles at $z = k\pi, k \in \mathbb{Z}$. Only $z = 0$ is inside the contour.

$$\operatorname{Res}\left[\frac{z-1}{\sin(z)}; 0\right] = \frac{0-1}{\sin'(0)} = \frac{-1}{\cos(0)} = -1$$

so $I = -2\pi i$.

8.

$$\oint_{|z|=1} \cot(z) dz$$

Simple poles at $z = k\pi, k \in \mathbb{Z}$. The only singularity inside the contour is $z = 0$.

$$\text{Res}[\cot(z); 0] = \text{Res}\left[\frac{\cos(z)}{\sin(z)}; 0\right] = \frac{\cos(0)}{\sin'(0)} = \frac{\cos(0)}{\cos(0)} = 1.$$

So $I = 2\pi i$.

9.

$$I = \oint_{|z|=2} \frac{1}{z^3(z+4)} dz$$

Singularities at $z = 0$ and $z = -4$. Only $z = 0$ is inside the contour so

$$I = 2\pi i \,\text{Res}\left[\frac{1}{z^3(z+4)}; 0\right].$$

But we can also write

$$I = \text{Res}\left[\frac{1}{z^3(z+4)}; \infty\right] - \text{Res}\left[\frac{1}{z^3(z+4)}; -4\right]$$

This is easier to evaluate because the residue at infinity is zero and $z = -4$ is a simple pole (compared to a pole of order 3 at $z = 0$). Thus

$$I = 2\pi i \left[0 - \frac{1}{(-4)^3}\right] = \frac{2\pi i}{64} = \frac{\pi i}{32}.$$
