Homework from Saff and Snider 6.7

filename: `ss6.7.tex`

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6.7:6

Let $q(z) = z^6 + 4z^2$ and $p(z) = z^6 + 4z^2 - 1 = q(z) - 1$. On the contour $|z| = 1$ we have

$$|q(z)| = |z|^2 \, |z^4 + 4| = |z^4 + 4| \geq 4 - |z|^4 = 4 - 1 = 3 > 1.$$

Thus Rouché's theorem says that q and p have the same number of zeros in the unit disk. The polynomial $q(z)$ has a zero of order 2 at $z = 0$. The other zeros of q lie on the circle $|z| = \sqrt{2}$. So q and therfore also p have 2 zeros (counted with multiplicity) in the unit disk.

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When $|z| = 2$ we have

$$|z^3 + 27| \geq 27 - 8 = 19 > 18 = |9z|$$

So Rouché's theorem says $z^3 + 27$ and $z^3 + 9z + 27$ have the same number of zeros in the disk of radius 2. But all the zeros of $z^3 + 27$ lie on the circle $|z| = 3$. Thus both $z^3 + 27$ and $z^3 + 9z + 27$ have no zeros in the disk of radius 2.

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We wish to show that all the roots of $p(z) = z^6 - 5z^2 + 10$ lie in the annulus $1 < |z| < 2$.

When $|z| = 2$ we have

$$|z^6 + 10| \geq |z|^6 - 10 = 64 - 10 = 54 > 20 = |5z^2|.$$

This shows that $z^6 + 10$ and $z^6 - 5z^2 + 10$ have the same number of zeros in $\{z : |z| < 2\}$. $\{z : |z| < 2\}$. The zeros of $z^6 + 10$ satisfy $|z| = (10)^{1/6}$ which is < 2. So $z^6 - 5z^2 + 10$ has 6 (i.e., all) zeros in $\{z : |z| < 2\}$.

On the other hand, when $|z| = 1$, then $|z^6 - 5z^2| < |z|^6 + 5|z|^2 = 6 < 10$. So $z^6 - 5z^2 + 10$ has the same number of zeros in $\{z : |z| = 1\}$ as the constant polynomial 10, i.e., none.

Thus the zeros must all lie in the annulus $1 < |z| < 2$.

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When $|z| = 1$ we have

$$|z^3 - 2z^2 + z - 1| \le |z|^3 + 2|z|^2 + |z| + 1 = 5 < 6 = |6z^4|.$$

This implies that $6z^4$ and $6z^4 + z^3 - 2z^2 + z - 1$

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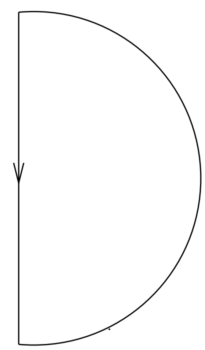
We want to find the winding number of $f(z) = 2 - e^{-z} - z$ around a D shaped contour consisting of a semi-circle in the right half plane of radius R and centre 0 together with the segment $[iR, -iR]$ on the imaginary axis.



When $z = iy$ with $y \in \mathbb{R}$ we have $\operatorname{Re} f(iy) = 2 - \cos(y) \ge 1$ so the direction vector $\hat{f}(iy) = f(iy)/|f(iy)|$ lies in the right half plane. This means that the winding can be at most π. Since the direction of $\hat{f}$ approaches $-i$ when $y \to \infty$ and i when $y \to -\infty$ we see that the change in argument when y travels down the imaginary axis is π. On the semicircle part we have $\hat{f}(Re^{-\theta}) \to -e^{i\theta}$ as $R \to \infty$. So the semi circle also contributes π. So the total winding is 2π and we conclude that the number of zeros is 1.

If z is a zero, i.e., $2 - e^z - z = 0$ then if taking the conjugate yields $2 - e^{\overline{z}} - \overline{z} = 0$ so that $\overline{z}$ is a solution whenever z is. If z were not real then we would have two distince solutions z and $\overline{z}$ contradicting our counting result.

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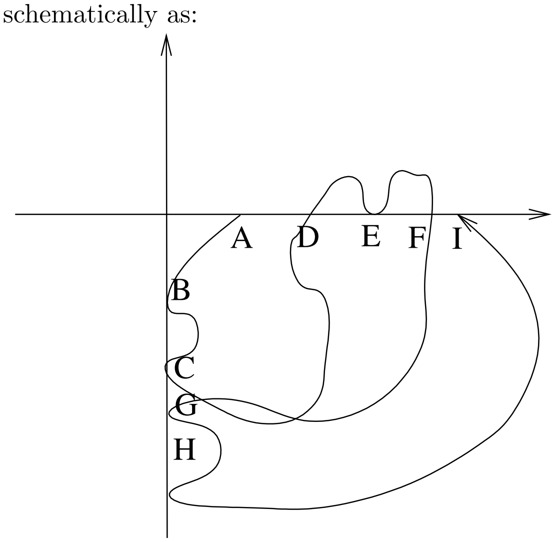
To use the Nyquist crieterion we must determine the change in argument in $p(iy)$ as y goes from ∞ to $-\infty$. Following the hint we write

$$p(iy) = (y^2 - 2)(y^2 - 1) + i(y(1 - 2y^2).$$

and find where p hits the axes. This will only occur if $\operatorname{Re} p(iy) = 0$ (at $y \in \{\pm 1, \pm\sqrt{2}\}$) or $\operatorname{Im} p(iy) = 0$ (at $y \in \{0, \pm 1/\sqrt{2}\}$). We collect the information about the crossings:

	y	$\mathrm{Re}(p(iy))$	$\mathrm{Im}(p(iy))$
A	$+\infty$	$y^4 + O(y^3)$	$O(y^3)$
B	$\sqrt{2}$	0	$\sqrt{2}(1 - 2 \cdot 2) < 0$
C	1	0	$1(1 - 2) < 0$
D	$1/\sqrt{2}$	$((1/2) - 2)(1/2 - 2) > 0$	0
E	0	$2 > 0$	0
F	$-1/\sqrt{2}$	$((1/2) - 2)(1/2 - 2) > 0$	0
G	-1	0	$1(1 - 2) < 0$
H	$-\sqrt{2}$	0	$\sqrt{2}(1 - 2 \cdot 2) < 0$
I	$-\infty$	$y^4 + O(y^3)$	$O(y^3)$

This can be viewed schematically as:



I haven't checked to make sure the line is on the right side of the axis e.g., between D and E because it doesn't affect the winding. From the picture we see there is no winding along the imaginary axis. So the total winding is 4π(from the semicircle) $+0$. So the number of zeros in the right half plane is $4\pi/(2\pi) = 2$.