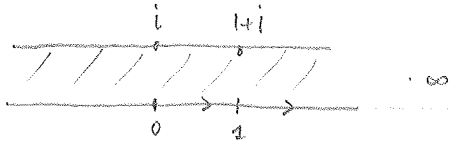
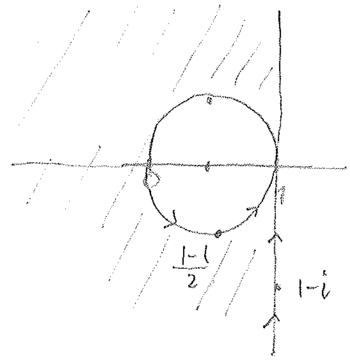
Math 301 Hmk7

7.3 #2: Find the image of the boundary first. The boundary consists of two lines: $\mathbb{R}$ and $\mathbb{R}+i$. Since FLT's maps circles/lines to circles/lines we need only check three points. $f(z) = (z-i)/z$ maps $0 \to \infty$, $1 \to 1-i$, $\infty \to 1$ so the image of $\mathbb{R}$ is a line through 1 and $1-i$. $f(z)$ maps $i \to 0$, $1+i \to \frac{1}{1+i} = \frac{1-i}{2} = \frac{1}{2} - \frac{i}{2}$ $\infty \to 1$, so the image is a circle centred at $\frac{1}{2}$ with radius $\frac{1}{2}$. Checking orientations, we can draw the image.





7.3 # 3: (a) this is a translation. The image is $|w-(2-2i)| = 1$

(b) this is a scaling $|z-2| = \left|\frac{3iz}{3i} - \frac{6i}{3i}\right| = \frac{1}{3}|w-6i|$

so $|z-2| = 1 \iff |w-6i| = 3$.

(c) The image is a circle or line. Check 3 points: $1 \to \infty$, $2+i \to \frac{i}{1+i} = \frac{i(1-i)}{2} = \frac{1+i}{2}$, $3 \to \frac{1}{2}$, so the image is a vertical line through $\frac{1}{2}$.

(d) $1 \to \frac{3}{2}$, $2+i \to \frac{-2+i}{-1+i} = \frac{(-2+i)(-1-i)}{2} = \frac{2+i+1}{2} = \frac{3}{2}+\frac{i}{2}$, $3 \to \infty$

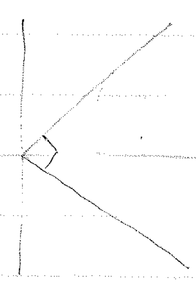
so the image is a vertical line through $\frac{3}{2}$.

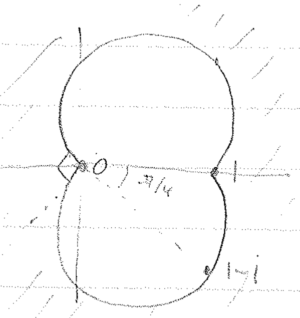
(e) $1 \to 1$, $2+i \to \frac{1}{2+i} = \frac{2-i}{5} = \frac{2}{5} - \frac{i}{5}$, $3 \to \frac{1}{3}$. By symmetry the image is a circle centred on the real axis so it must be $|z - \frac{2}{3}| = \frac{1}{3}$.

7.3 #6 Let $z_1\ z_2\ z_3$ be fixed points of a F.L.T. $f(z)$. We know there is a unique FLT $g(z)$ mapping $z_1 \to 0$, $z_2 \to 1$ and $z_3 \to \infty$

(Pf: Let $g(z) = \frac{az+b}{cz+d}$ then $g(z_1) = 0 \Rightarrow az_1 + b = 0$ and $g(z_3) = \infty$ $\Rightarrow cz_3 + d = 0$, so $g(z) = \frac{az - az_1}{cz - cz_3} = \left(\frac{a}{c}\right)\left(\frac{z-z_1}{z-z_3}\right)$. Now $g(z_2) = 1 \Rightarrow$ $\left(\frac{a}{c}\right) = \frac{z_2 - z_3}{z_1 - z_3}$ so $g(z)$ is determined.) Now $g \circ f(z)$ also sends $z_1 \to 0$, $z_2 \to 1$, $z_3 \to \infty$, so $g \circ f(z) = g(z)$. Apply $g^{-1}(z)$ to both sides to conclude $f(z) = z$.

7.3 #9 Under the map $w = z/(z-1)$ $0 \to 0$, $\infty \to 1$ so the images of the boundary rays are arcs of circles through 0 and 1. Note $1+i \to 1-i$ so the image looks like this





How can we find the centre of the circle? The brute force method would be to solve $|a-0|^2 = |a-1|^2 = |a-1+i|^2$ for a. But we could also note the by conformality, the line tangent to the circle at 0 makes an angle of $-3\pi/4$ with the x axis, so the diameter makes an angle of $-\pi/4$. So the midpt is $\frac{1-i}{2}$

and the radius is $\frac{|1-i|}{2} = \frac{1}{\sqrt{2}}$, i.e. the circle is $|z - \frac{1-i}{2}| = \frac{1}{\sqrt{2}}$

The upper circle will be $|z - \frac{1+i}{2}| = \frac{1}{\sqrt{2}}$.

7.3 # 11 Use the map $f(z) = \frac{4\pi i z}{z-2}$, Then $2 \to \infty$, $0 \to 0$, $-2 \to 2\pi i$

Since the circles only intersect at 2, the images will be parallell lines through 0 and $2\pi i$, Since $f(-1) = \frac{4}{3}\pi i$ the image lies between the lines. Thus $\exp(z)$ will map this region to a strip.

7.4 # 4 Let $f(z) = (z, z_2, z_3, z_4)$ then $f \circ T^{-1}$ maps

$T(z_2) \to 0$, $T(z_3) \to 1$, $T(z_4) \to \infty$. Thus $f(T^{-1}(z)) = (z, T(z_2), T(z_3), T(z_4))$

Now set $z = T(z_1)$

7.4 # 8 If S and T both map $z_1 \to w_1$, $z_2 \to w_2$ $z_3 \to w_3$ then

$S^{-1} \circ T$ fixes z_1, z_2, z_3. Thus $S^{-1} \circ T(z) = z$ (from previous

problem, so $T(z) = S(z)$.

7.4 # 9 Since $1, -1$ are symmetric w.r.t the imaginary axis, $f(1) = \infty$ and $f(-1)$ are symmetric w.r.t the unit circle, Thus $f(-1) = 0$.

1. $$f_{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}\left(f_{\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}}(z) \right) = f_{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}\left(\frac{a'z+b'}{c'z+d'} \right)$$

$$= \frac{a(a'z+b') + b(c'z+d')}{c(a'z+b') + d(c'z+d')} = \frac{(aa'+bc')z + ab'+bd'}{(ca'+cb')z + cb'+dd'}$$

$$= f_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}}(z)$$

2. $$f_{\alpha\begin{pmatrix} a & b \\ c & d \end{pmatrix}}(z) = \frac{\alpha a z + \alpha b}{\alpha c z + \alpha d} = \frac{az+b}{cz+d} = f_{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}(z).$$

3. Since $f_{A^{-1}} \circ f_A(z) = f_{A^{-1}A}(z) = f_I(z) = z$,

$f_{A^{-1}} = (f_A)^{-1}$. Since $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$,

$$f_{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}^{-1} = f_{\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}} = f_{\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}.$$