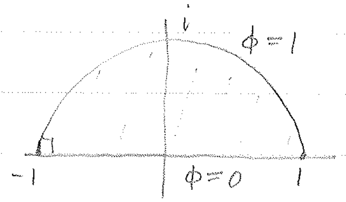
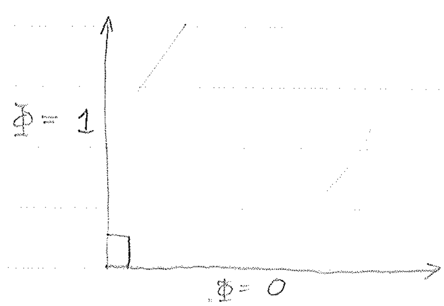
$\boxed{\text{Math 307 Hmk 8}}$

$\boxed{7.6 \# 1}$ We must solve Laplace's equation with B.C.



Let use a FLT that maps $-1 \to 0$, $0 \to 1$, $1 \to \infty$

$f(z) = -\frac{z+1}{z-1}$. Then $[-1, 1]$ maps to the ray $[0, \infty)$. The semicircle boundary also maps to a ray from 0 to ∞. We can use conformality to see it must be the positive imaginary axis. Thus the image is.

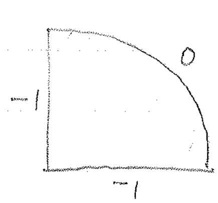


In this domain, the solution is $\Phi(u,v) = \frac{2}{\pi} \operatorname{Arg}(u+iv) = \frac{2}{\pi} \tan^{-1}\left(\frac{v}{u}\right)$

$$f(x+iy) = -\frac{(x+iy+1)(x-iy-1)}{(x-1)^2+y^2} = \underbrace{\frac{1-x^2-y^2}{(x-1)^2+y^2}}_{u(x,y)} + i\underbrace{\frac{2y}{(x-1)^2+y^2}}_{v(x,y)}$$

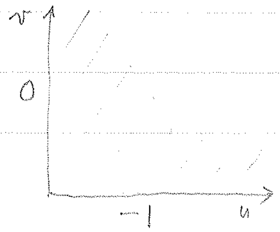
So $\phi(x,y) = \Phi(u(x,y), v(x,y)) = \frac{2}{\pi} \tan^{-1}\left(\frac{2y}{1-x^2-y^2}\right)$

$\boxed{7.6 \# 3}$ We need to solve Laplace's equation in the region.



Notice that the map $z \mapsto z^2$ maps this region into the one from problem #1

Thus $f(z) = -\dfrac{z^2+1}{z^2-1}$ will map the given region into



The solution in this domain is

$$\Phi(u,v) = \frac{2}{\pi}\,\text{Arg}(u+iv) - 1$$

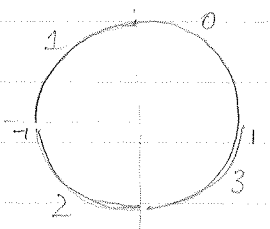
$$f(z) = -\frac{(z^2+1)(\bar{z}^2-1)}{|z^2-1|^2} = -\frac{\left(|z|^4 - (z^2-\bar{z}^2) - 1\right)}{|z^2-1|^2}$$

$$= -\frac{\left(|z|^4 - 2i\,\text{Im}(z^2) - 1\right)}{|z^2-1|^2} = \underbrace{\frac{1-(x^2+y^2)^2}{|z^2-1|^2}}_{u(x,y)} + i\,\underbrace{\frac{4xy}{|z^2-1|^2}}_{v(x,y)}$$

So

$$\phi(x,y) = \Phi(u(x,y), v(x,y)) = \frac{2}{\pi}\tan^{-1}\left(\frac{4xy}{1-(x^2+y^2)^2}\right) - 1$$

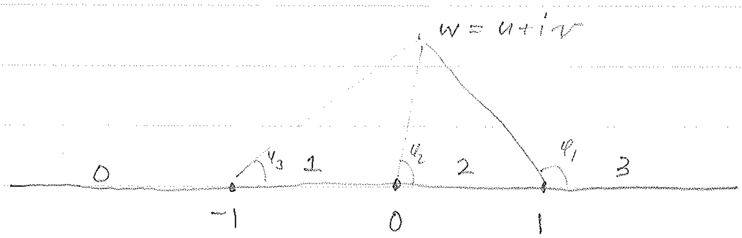
$\boxed{7.6\ \#4}$ We need to solve Laplace's equation in



Map to the upper half plane via

$f(z) = -i\,\dfrac{z+1}{z-1}$. Then $f(1) = \infty$, $f(i) = -1$,

$f(-1) = 0$, $f(-i) = 1$. The image region is has bounding values as drawn below:



Soln is given by

$\Phi = c_0 + c_1\varphi_1 + c_2\varphi_2 + c_3\varphi_3$ where

$c_0, \dots, c_3$ are constants chosen to satisfy B.C. So we choose

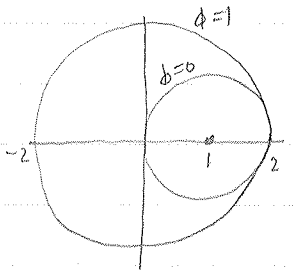
$c_0 = 3$, $c_1 = -\frac{1}{\pi}$, $c_2 = -\frac{1}{\pi}$, $c_3 = -\frac{1}{\pi}$. Since $v > 0$ in the region it is most convenient to write $\varphi_1 = \cot^{-1}\left(\frac{u-1}{v}\right)$, $\varphi_2 = \cot^{-1}\left(\frac{u}{v}\right)$, $\varphi_3 = \cot^{-1}\left(\frac{u+1}{v}\right)$. We have

$$f(x+iy) = -i\,\frac{(x+iy+1)(x-iy-1)}{(x-1)^2+y^2} = \frac{-i\left[(x+1)(x-1)+y^2-2iy\right]}{(x-1)^2+y^2}$$

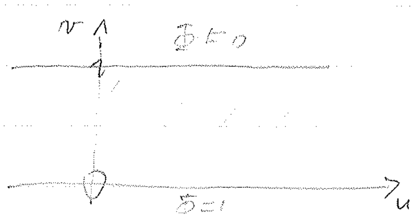
$$= \underbrace{\frac{-2y}{(x-1)^2+y^2}}_{u(x,y)} + i\,\underbrace{\frac{1-x^2-y^2}{(x-1)^2+y^2}}_{v(x,y)}$$

$$\phi(x,y) = 3 - \frac{1}{\pi}\cot^{-1}\left(\frac{-2y-(x-1)^2-y^2}{1-x^2-y^2}\right) - \frac{1}{\pi}\cot^{-1}\left(\frac{-2y}{1-x^2-y^2}\right) - \frac{1}{\pi}\cot^{-1}\left(\frac{-2y+(x-1)^2+y^2}{1-x^2-y^2}\right)$$

$\boxed{7.6\ \#\ 6}$ The region can be mapped to a strip by a FLT that maps $2 \to \infty$. Let's map $-2 \to 0$, $0 \to i$, $2 \to \infty$ with



$$f(z) = -i\,\frac{z+2}{z-2}$$ Then the image is



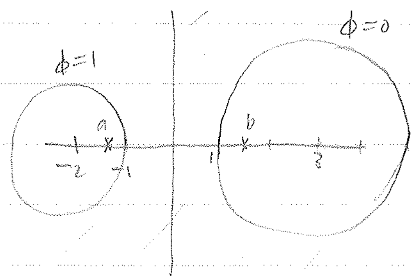
So the solution is $\Phi = 1 - v$

$$f(z) = -i\,\frac{(x+iy+2)(x-2-iy)}{(x-2)^2+y^2}$$

$$= \frac{-i\left[(x^2-2+y^2)-4iy\right]}{(x-2)^2+y^2}$$

$$= \underbrace{\frac{-4y}{(x-2)^2+y^2}}_{u(x,y)} + i\,\underbrace{\frac{(4-x^2-y^2)}{(x-2)^2+y^2}}_{v(x,y)} \qquad \phi(x,y) = 1 - \frac{(4-x^2-y^2)}{(x-2)^2+y^2}$$

$\boxed{7.6 \ \# 8}$



Find points so that both circles are symmetric w.r.t. them: By symmetry, the points a, b will lie on the real axis, and

$$(3-b)(3-a) = 2^2 = 4$$
$$(a+2)(b+2) = 1^2 = 1$$

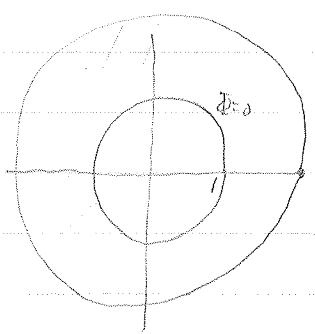
$b+2 = \frac{1}{a+2}$, $b = \frac{1}{a+2} - 2$, substitute into first eqn $3-b = 5 - \frac{1}{a+2}$

$(5 - \frac{1}{a+2})(3-a) = 4 \qquad 5a^2 - 2a - 19 = 0 \qquad a = \frac{1-4\sqrt{6}}{5} \qquad b = \frac{1+4\sqrt{6}}{5}$

Now we use a FLT that takes $b \to 0$, $1 \to 1$, $a \to \infty$

$$f(z) = \frac{(z-b)}{(z-a)}\frac{1-a}{1-b} = \frac{1-a}{1-b}\frac{(x+iy-b)(x-iy-a)}{(x-a)^2+y^2} = \left(\frac{1-a}{1-b}\right)\frac{(x-b)(x-a)+y^2}{(x-a)^2+y^2}$$
$$+ i\left(\frac{1-a}{1-b}\right)\frac{-(a+b)y}{(x-a^2)+y^2}$$
$$= u(x,y) + i v(x,y).$$

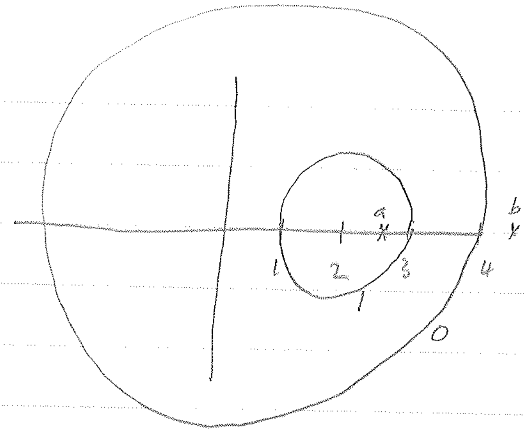
The image is two concentric circles



$\frac{(1+b)(1-a)}{(1+a)(1-b)} = 5 + 2\sqrt{6} \qquad \Phi(u,v) = \frac{1}{\ln(5+2\sqrt{6})}\ln\left(\sqrt{u^2+v^2}\right)$

$$\phi(x,y) = \frac{1}{\ln(5+2\sqrt{6})}\ln\left(\sqrt{\left(\frac{1-a}{1-b}\right)^2 \frac{((x-b)(x-a)+y^2)^2 + (a+b)^2y^2}{(x-a)^2+y^2}}\right)$$

$\boxed{7.6 \ \# 9}$



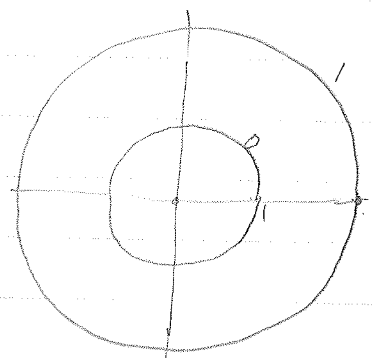
The symmetric points a, b lie on the real line and satisfy.

$$\left.\begin{aligned} (a-2)(b-2) &= 1 \\ ab &= 16 \end{aligned}\right\} \quad (a-2)\left(\frac{16}{a}-2\right) = 1$$

This has solution $\quad a = \dfrac{19-\sqrt{105}}{4} \qquad b = \dfrac{19+\sqrt{105}}{4}$

map $\quad b \to 0, \ 4 \to 1, \ a \to \infty \quad$ using $\quad f(z) = \left(\dfrac{z-b}{z-a}\right)\left(\dfrac{4-a}{4-b}\right)$

Then the image is



$$\left(\frac{3-b}{3-a}\right)\left(\frac{4-a}{4-b}\right)$$

Then the solution is $\quad \dfrac{1}{\ln\left(\dfrac{(3-b)}{(3-a)}\cdot\dfrac{(4-a)}{(4-b)}\right)} \ln\left(|f(z)|\right).$