$$\frac{(w-i+1)}{(w-i-1)}\frac{(i-1)}{(i+1)} = \frac{-iz+i}{z+1}$$

$$f_{\begin{pmatrix} i & 1+i \\ 1 & -(1+i)\end{pmatrix}}(w) = f_{\begin{pmatrix} -i & i \\ 1 & 1\end{pmatrix}}(z) \qquad w = f_{\begin{pmatrix} -(1+i) & -(1+i) \\ -1 & i\end{pmatrix}\begin{pmatrix} -i & i \\ 1 & 1\end{pmatrix}}(z)$$

$$f = f_{\begin{pmatrix} i-1-1-i & -i+1-i-1 \\ 2i & 0\end{pmatrix}}(z) = \frac{-2z-2i}{2iz} = \frac{-z-i}{iz} = \frac{iz-1}{z}$$

$$f_{\begin{pmatrix} -2 & -2i \\ 2i & 0\end{pmatrix}}(z)$$

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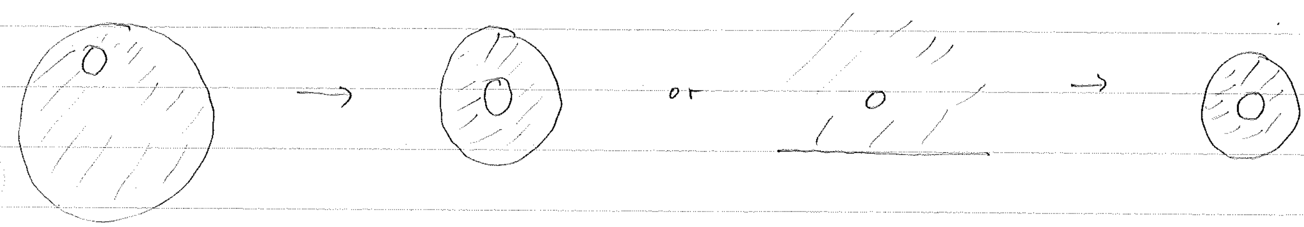
## Lecture 21

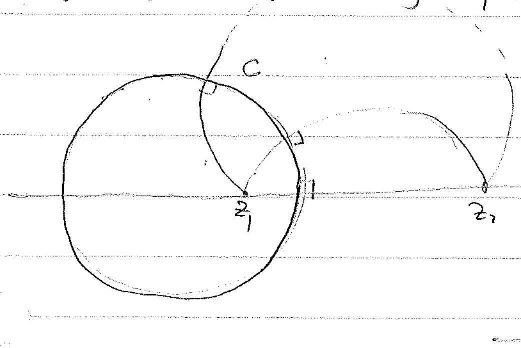
**Goal:** Find a FLT that maps the region between 2 circles / lines to an annulus.



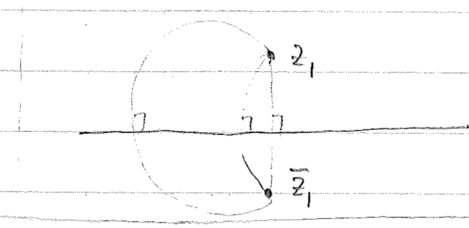
Note we can't specify, e.g. three pts on each circle...

### Symmetric points

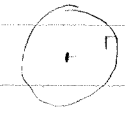
**Defn** $z_1$ and $z_2$ are symmetric w.r.t. a circle $C$ if every line or circle connecting $z_1, z_2$ meets $C$ orthogonally.




Special case: if $C = \mathbb{R}$, $z_1$ and $\bar{z}_1$ are symmetric




Special case: if $z_1$ is the centre of $C$ then $z_2 = \infty$



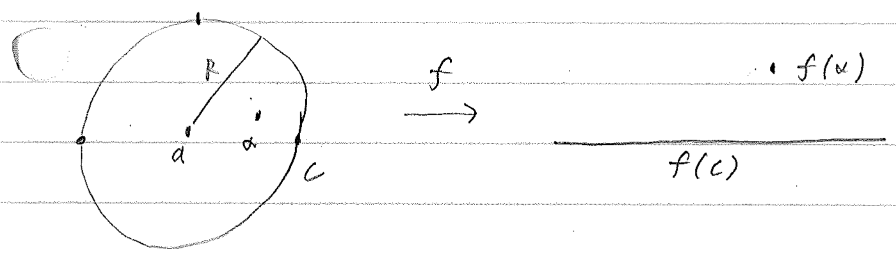
<u>Mapping of symmetric points</u> : If $z_1$, $z_2$ are symmetric w.r.t. $C$ and $f$ is a FLT then $f(z_1)$ and $f(z_2)$ are symm. w.r.t. $f(C)$.

Follows from the fact that FLT map circles/lines $\to$ circles/lines and is conformal.

<u>Finding symmetric points</u> :

Given a circle $C = \{ z : |z-a| = R \}$ and a point $\alpha$ what point $\alpha^*$ is symmetric to $\alpha$ w.r.t. $C$?

<u>Strategy</u> : • find a FLT mapping $C$ to $\mathbb{R}$




Then $\overline{f(\alpha)}$ and $f(\alpha)$ are symm. w.r.t. $f(C) = \mathbb{R}$. Thus

$f^{-1}(\overline{f(\alpha)})$ and $f^{-1}(f(\alpha)) = \alpha$ are sym w.r.t. $f^{-1}(\mathbb{R}) = C$

To find $f$ we pick 3 pts on $C$ : $a-R$, $a+iR$, $a+R$
Then $(z, a-R, a+iR, a+R)$ maps these pts to $0, 1, \infty$, so it maps $C$ to $\mathbb{R}$.

$$f(z) = (z, a-R, a+iR, a+R) = \left( \frac{z-(a-R)}{z-(a+R)} \right) \left( \frac{a+iR-(a+R)}{a+iR-(a-R)} \right)$$

$$= \frac{z-(a-R)}{z-(a+R)} \frac{(iR-R)}{(iR+R)} = \frac{iz - i(a-R)}{z-(a+R)} \quad \left( = f_{\begin{pmatrix} i & -i(a-R) \\ 1 & -(a+R) \end{pmatrix}}(z) \right)$$

4. glasses water ... of the

$$f(\alpha) = \frac{i\alpha - i(a-R)}{\alpha - (a+R)}$$

$$\overline{f(\alpha)} = \frac{-i\bar{\alpha} + i(\bar{a}-R)}{\bar{\alpha} - (\bar{a}+R)}$$

$$\alpha^* = f^{-1}\left(\overline{f(\alpha)}\right) = f_{\begin{pmatrix} -(a+R) & i(a-R) \\ -1 & i \end{pmatrix}}\left(\overline{f(\alpha)}\right) = f_{\begin{pmatrix} -(a+R) & i(a-R) \\ -1 & i \end{pmatrix}} f_{\begin{pmatrix} -i & i(\bar{a}-R) \\ 1 & -(\bar{a}+R) \end{pmatrix}}(\bar{\alpha})$$

$$= \frac{-(a+R)\left(\dfrac{-i\bar{\alpha}+i(\bar{a}-R)}{\bar{\alpha}-(\bar{a}+R)}\right) + i(a-R)}{\dfrac{i\bar{\alpha}-i(\bar{a}-R)}{\bar{\alpha}-(\bar{a}+R)} + i}$$

$$\begin{pmatrix} -(a+R) & i(a-R) \\ -1 & i \end{pmatrix}\begin{pmatrix} -i & i(\bar{a}-R) \\ 1 & -(\bar{a}+R) \end{pmatrix}$$

$$= \begin{pmatrix} i(a+R)+i(a-R) & -i(a+R)(\bar{a}-R)-i(a-R)(\bar{a}+R) \\ i+i & -i(\bar{a}-R)-i(\bar{a}+R) \end{pmatrix}$$

$$= \begin{pmatrix} 2ia & -i(2|a|^2+2R^2) \\ 2i & -2i\bar{a} \end{pmatrix}$$

$$\to \frac{a\bar{\alpha} - (|a|^2+R^2)}{\bar{\alpha} - \bar{a}}$$

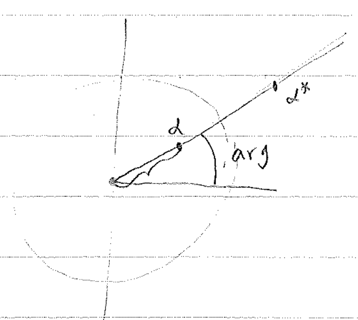
$$= \frac{(a+R)\bar{\alpha} - (a+R)(\bar{a}-R) + (a-R)\bar{\alpha} - (a-R)(\bar{a}+R)}{\bar{\alpha} - (\bar{a}-R) + \bar{\alpha} - (\bar{a}+R)}$$

$$= \frac{2a\bar{\alpha} - |a|^2 + \cancel{aR} - \cancel{\bar{a}R} + R^2 - |a|^2 - \cancel{aR} + \cancel{\bar{a}R} + R^2}{2\bar{\alpha} - 2\bar{a}}$$

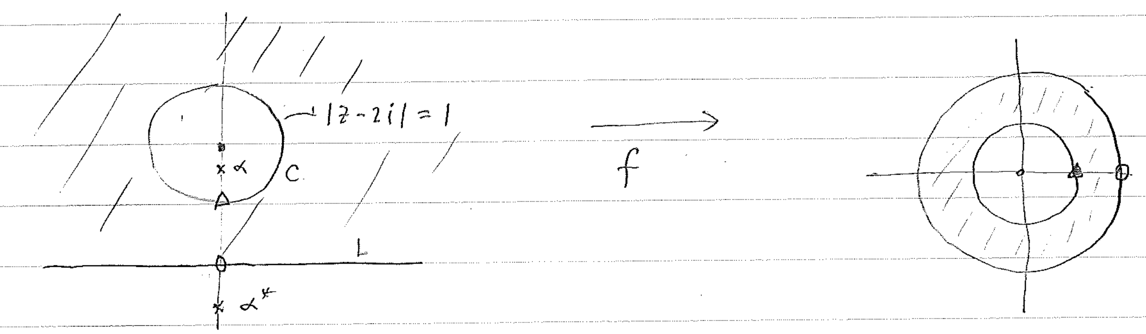
$$= \frac{a\bar{\alpha} - |a|^2 + R^2}{\bar{\alpha} - \bar{a}} = \frac{a(\bar{\alpha}-\bar{a}) + R^2}{\bar{\alpha} - \bar{a}} = \boxed{a + \frac{R^2}{\bar{\alpha}-\bar{a}} = \alpha^*}$$

Interpretation $\quad \alpha^* - a = \dfrac{R^2(\alpha - a)}{|\alpha - a|^2} \Rightarrow \arg(\alpha^* - a) = \arg(\alpha - a)$

$$|\alpha^* - a| = \frac{R^2}{|\alpha - a|}$$



**Example** Find a FLT mapping $D$ onto an annulus




**Step 1** find $\alpha, \alpha^*$ that are symmetric w.r.t. both $C, L$.

**Step 2** find a FLT mapping $\alpha \to 0$ and $\alpha^* \to \infty$

Then $f(C)$ and $f(L)$ must be symmetric w.r.t. $0$ and $\infty$

i.e. they are circles centred at $0$.

**Step 1**: $\alpha^*, \alpha$ are symm. w.r.t. $L$ if $\alpha^* = \bar{\alpha}$

$\alpha^*, \alpha$ are symm. w.r.t. $C$ $(a = 2i,\ R = 1)$ if $\alpha^* = a + \dfrac{R^2}{\bar{\alpha} - \bar{a}}$

$$= 2i + \frac{1}{\bar{\alpha} + 2i}$$

So solve $\bar{\alpha} = 2i + \dfrac{1}{\bar{\alpha} + 2i}$ $\quad (\bar{\alpha} - 2i)(\bar{\alpha} + 2i) = 1 \quad (\bar{\alpha})^2 + 4 = 1$

$\bar{\alpha}^2 = -3 \quad \bar{\alpha} = \pm i\sqrt{3} \quad \alpha = i\sqrt{3} \quad \alpha^* = -i\sqrt{3}$

**Step 2** We want $\alpha \to 0$, $\alpha^* \to \infty$ One more choice say $0 \to 1$

$\cancel{(w, 0, 1, \infty)}\ (z, i\sqrt{3}, 0, -i\sqrt{3}) = w$

$$w = -\frac{z - i\sqrt{3}}{z + i\sqrt{3}}$$

$z = x \Rightarrow |w| = 1$ ✓

$z = 2i + e^{i\theta} \quad w = -\dfrac{(2-\sqrt{3})i + e^{i\theta}}{(2+\sqrt{3})i + e^{i\theta}}$

$|w| = [(2-\sqrt{3})i + e^{i\theta}][-(2+\sqrt{3})i + e^{-i\theta}]$ hard.

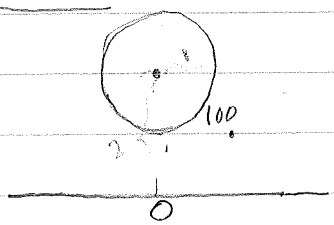
Then $f(L) = \{ |w| = 1 \}$

$f(C) = ?$ well, $f(i) = -\dfrac{i - i\sqrt{3}}{i + i\sqrt{3}} = \dfrac{\sqrt{3}-1}{\sqrt{3}+1} = \dfrac{(\sqrt{3}-1)^2}{2} = \dfrac{4-2\sqrt{3}}{2} = 2-\sqrt{3}$

So it must be $\{ |w| = 2-\sqrt{3} \}$.

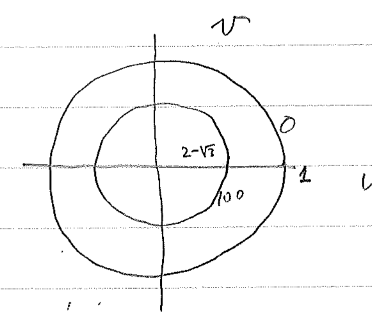
Lecture 22

Problem: Calculate the steady state temperature distn $t(x,y)$ (eg a pipe with boiling water travelling throug grond. What it $t(1+i)$?




Need to solve $\Delta t = 0$ with B.C. as shown.

Soln: map via $f(z) = -\dfrac{z - i\sqrt{3}}{z + i\sqrt{3}}$ to




$T(u,v) = A \ln\left(\sqrt{u^2+v^2}\right) + B$

$\left( = \mathrm{Re}\left( A \log(w) + B \right) \right) \leftarrow$ explain: even though log has a branch cut

$T(2-\sqrt{3}, 0) = 100 \qquad T(1,0) = 0$

$A \ln(2-\sqrt{3}) + B = 100 \qquad A \log(1) + B = 0$

$\Rightarrow \quad B = 0 \quad A = \dfrac{100}{\ln(2-\sqrt{3})}$

$T(u,v) = \dfrac{100}{\ln(2-\sqrt{3})} \ln\left(\sqrt{u^2+v^2}\right) = \dfrac{50}{\ln(2-\sqrt{3})} \ln(u^2+v^2)$

$f(z) = \dfrac{-x - iy + i\sqrt{3}}{x + i(y+\sqrt{3})} = \dfrac{\left(-x + i(-y+\sqrt{3})\right)\left(x - i(y+\sqrt{3})\right)}{x^2 + (y+\sqrt{3})^2}$

$= \dfrac{-x^2 + 3 - y^2}{x^2 + (y+\sqrt{3})^2} + i\dfrac{\left( x(y+\sqrt{3}) + x(-y+\sqrt{3}) \right)}{x^2 + (y+\sqrt{3})^2}$

$$= \underbrace{\frac{3 - x^2 - y^2}{x^2 + (y+\sqrt{3})^2}}_{u(x,y)} + i \underbrace{\frac{2\sqrt{3}\,x}{x^2 + (y+\sqrt{3})^2}}_{v(x,y)}$$

$$t(x,y) = T(u(x,y), v(x,y)) = \frac{100}{\ln(2-\sqrt{3})} \frac{1}{2} \ln\left( u^2 + v^2 \right)$$

$$= \frac{50}{\ln(2-\sqrt{3})} \ln\left( \frac{x^2 + (y-\sqrt{3})^2}{x^2 + (y+\sqrt{3})^2} \right)$$

$$t(1,1) = \frac{50}{\ln(2-\sqrt{3})} \ln\left( \frac{1 + (1-\sqrt{3})^2}{1 + (1+\sqrt{3})^2} \right) \cong 64.797^\circ$$