Math 301 Lecture 20

(*) examples on previous page

Recall: A FLT: $f(z) = \frac{az+b}{cz+d}$ $\quad ad-bc \neq 0$

maps circles/lines to circles/lines

ex

Goal: given $\underbrace{z_1\ z_2\ z_3}_{\text{distinct}}$ $\underbrace{w_1\ w_2\ w_3}_{\text{distinte}}$ find a FLT that maps

$z_i \to w_i \quad i = 1,2,3$

Subgoal find FLT that maps $z_1 \to 0$, $z_2 \to 1$, $z_3 \to \infty$

$$f(z) = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} \quad \text{called the cross ratio}$$

$$= (z, z_1, z_2, z_3)$$

$$= \lim_{z_1 \to \infty} \frac{\left(\frac{z}{z_1}-1\right)(z_2-z_3)}{(z-z_3)\left(z_2/z_1-1\right)}$$

Note: $(z, \infty, z_2, z_3) = \frac{(z_2-z_3)}{(z-z_3)}$, $(z, z_1, \infty, z_3) = \frac{(z-z_1)}{(z-z_3)}$

$$(z, z_1, z_2, \infty) = \frac{(z-z_1)}{(z_2-z_1)}$$

Back to main goal If $f(z) = (z, z_1, z_2, z_3)$ is a FLT mapping $z_1 \to 0$ $z_2 \to 1$ $z_3 \to \infty$

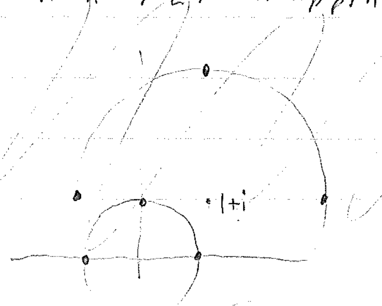
$g(w) = (w, w_1, w_2, w_3)$ $\quad w_1 \to 0$ $w_2 \to 1$ $w_3 \to \infty$

Then $w = g^{-1} \circ f(z)$ is the desired transfn. To compute

$$\begin{array}{ccccc} & f & & g & \\ z_1 & \to & 0 & \leftarrow & w_1 \\ z_2 & \to & 1 & \leftarrow & w_2 \\ z_3 & & \infty & \leftarrow & w_3 \end{array}$$

equate $(w, w_1, w_2, w_3) = (z, z_1, z_2, z_3)$ and solve for w

Example: Find a FLT mapping $\{|z| = 1\}$ to $\{|z-(1+i)| = 2\}$

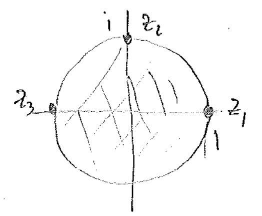


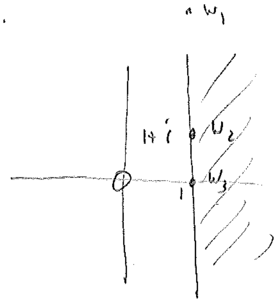
- pick 3 points on each circle eg

$z_1 = 1,\ z_2 = i,\ z_3 = -1$

$w_1 = 3+i,\ w_2 = 1+3i,\ w_3 = -1+i$

<u>Example</u>: Find a FLT that maps the unit circle to the half plane $\{ \operatorname{Re} z \geq 1 \}$.





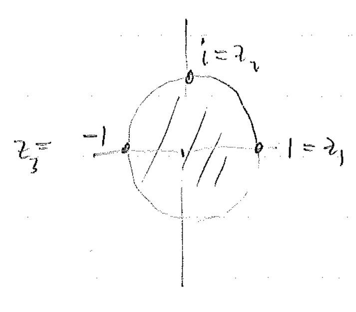
- pick points $z_1\ z_2\ z_3$ on the boundary of unit circle e.g. $1, i, -1$
- pick points $w_1\ w_2\ w_3$ on boundary of half plane with matching orientation (i.e. circle is on left when going $z_1 \to z_2 \to z_3$, so we want half plane on left when we go $w_1 \to w_2 \to w_3$) eg $\infty, i+1, 1$
- solve $(w, \infty, i+1, 1) = (z, 1, i, -1)$ for w

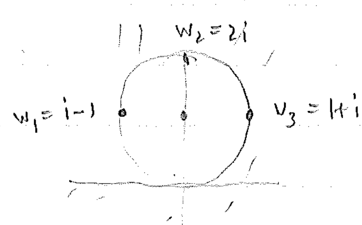
$$\frac{i}{w-1} = \left(\frac{z-1}{z+1}\right)\left(\frac{i+1}{i-1}\right) = \frac{-iz+i}{z+1}$$

$$f_{\begin{pmatrix} 0 & i \\ 1 & -1 \end{pmatrix}}(w) = f_{\begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}}(z)$$

$$w = f_{\begin{pmatrix} -1 & -i \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}}(z) = f_{\begin{pmatrix} 0 & -2i \\ i & -i \end{pmatrix}}(z) = \boxed{\frac{-2}{z-1}}$$

<u>Example</u> find a FLT that maps unit circle to <u>exterior</u> of $\{|z-i|=1\}$





send $1 \to i-1$
$i \to 2i$
$-1 \to i+1$

$\frac{z-1}{z+1}\ \frac{i+1}{i-1}$

solve

$(w, i-1, 2i, i+1) = (z, 1, i, -1)$ for w

$$f_{\underbrace{\begin{bmatrix} -1+i, & 2i \\ 1+i, & -2i \end{bmatrix}}_{A}}(w) = f_{\underbrace{\begin{bmatrix} 1+i & -1-i \\ -1+i & -1+i \end{bmatrix}}_{B}}$$

$A^{-1}B = \begin{bmatrix} 1 & i \\ -i & 0 \end{bmatrix} \to \frac{z-i}{-iz} = \frac{iz-1}{z}$

$\begin{matrix} i+1 & 2 \\ i-1 & 2 \end{matrix} \qquad \begin{matrix} i+1 & -(i+1) \\ i-1 & i-1 \end{matrix}$

$\begin{pmatrix} 2 & -2 \\ -i+1 & i+1 \end{pmatrix}\begin{pmatrix} i+1 & -(i+1) \\ i-1 & i-1 \end{pmatrix} = \begin{matrix} 2i+2-2i+2 \\ 2- \end{matrix}$

$$\boxed{\frac{(w-i+1)(i-1)}{(w-i-1)(i+1)} = \frac{-iz+i}{z+1}}$$

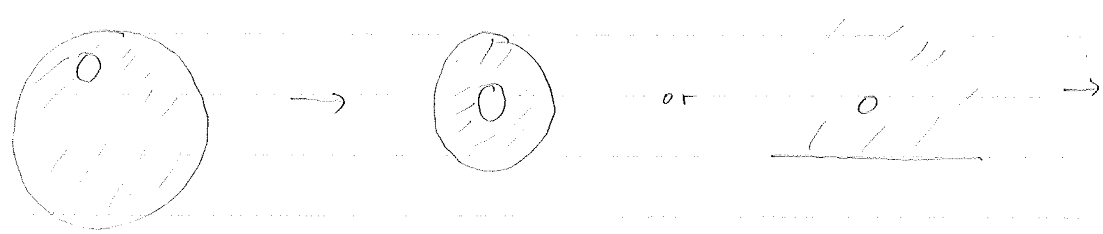
$$\boxed{f_{\begin{pmatrix} i & 1+i \\ 1 & -(1+i) \end{pmatrix}}(w) = f_{\begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}}(z)} \qquad \boxed{w = f_{\begin{pmatrix} -(1+i) & -(1+i) \\ -1 & i \end{pmatrix}\begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}}(z)}$$

$$f = f_{\begin{pmatrix} i+1-1-i & -i+1-i-1 \\ 2i & 0 \end{pmatrix}}(z) = \frac{-2z-2i}{2iz} = \frac{-z-i}{iz} = \frac{iz-1}{z}$$

$f_{\begin{pmatrix} -2 & -2i \\ 2i & 0 \end{pmatrix}}(z)$ ANS. $\boxed{\frac{iz-1}{z}}$

Lecture 21

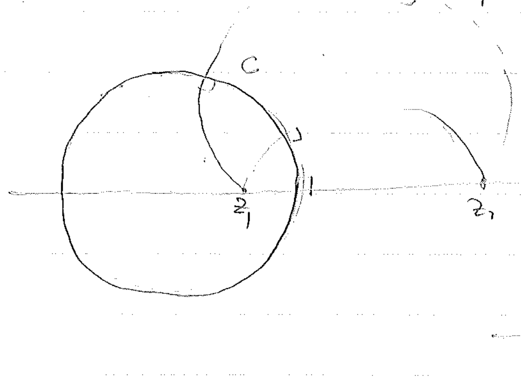
Goal: Find a FLT that maps the region between 2 circles / lines to an annulus.



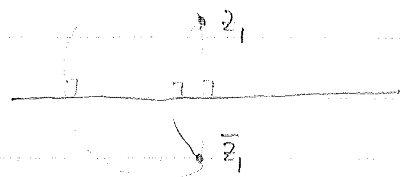
Note we can't specify, e.g. three pts on each circle...

Symmetric points

Defn z_1 and z_2 are symmetric w.r.t. a circle C if every line or circle connecting z_1, z_2 meets C orthogonally



Special case: if $C = \mathbb{R}$ z_1 and $\bar{z}_1$ are symmetric



Special case if z_1 is the center of C then $z_2 = \infty$

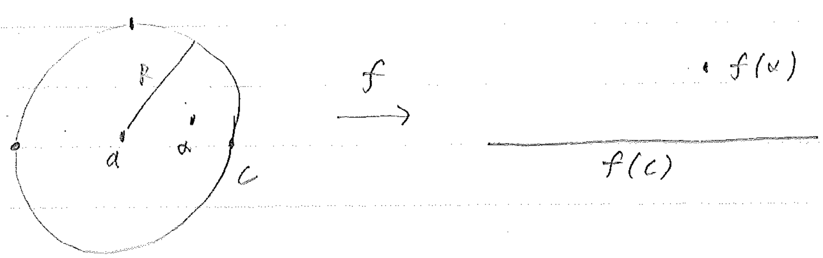
<u>Mapping of symmetric points</u> : If z_1, z_2 are symmetric w.r.t. C and f is a FLT then $f(z_1)$ and $f(z_2)$ are symm. w.r.t. $f(C)$.

Follows from the fact that FLT map circles/lines $\to$ circles/lines and is conformal.

<u>Finding symmetric points</u> :

Given a circle $C = \{ z : |z-a| = R \}$ and a point α what point α^* is symmetric to α w.r.t. C?

<u>Strategy</u> : • find a FLT mapping C to $\mathbb{R}$



Then $\overline{f(\alpha)}$ and $f(\alpha)$ are symm. w.r.t. $f(C) = \mathbb{R}$. Thus

$f^{-1}(\overline{f(\alpha)})$ is and $f^{-1}(f(\alpha)) = \alpha$ are sym w.r.t. $f^{-1}(\mathbb{R}) = C$

To find f we pick 3 pts on C : $a-R$, $a+iR$, $a+R$
Then $(z, a-R, a+iR, a+R)$ maps these pts to $0, 1, \infty$, so it maps C to $\mathbb{R}$.

$$f(z) = (z, a-R, a+iR, a+R) = \left(\frac{z-(a-R)}{z-(a+R)} \right) \left(\frac{a+iR-(a+R)}{a+iR-(a-R)} \right)$$

$$= \frac{z-(a-R)}{z-(a+R)} \underbrace{\frac{(iR-R)}{(iR+R)}}_{=i} = \frac{iz - i(a-R)}{z-(a+R)} \quad \left(= f_{\begin{pmatrix} i & -i(a-R) \\ 1 & -(a+R) \end{pmatrix}}(z) \right)$$

4 glasses water [illegible]

$$f(\alpha) = \frac{i\alpha - i(a-R)}{\alpha - (a+R)}$$

$$\overline{f(\alpha)} = \frac{-i\bar{\alpha} + i(\bar{a} - R)}{\bar{\alpha} - (\bar{a} + R)} = f_{\begin{bmatrix} -i & i(\bar{a}-R) \\ 1 & -(\bar{a}+R) \end{bmatrix}}(\bar{\alpha})$$

$$\alpha^* = f^{-1}(\overline{f(\alpha)}) = f_{\begin{pmatrix} -(a+R) & i(a-R) \\ -1 & i \end{pmatrix}}\left(\overline{f(\alpha)}\right) = f_{\begin{pmatrix} -(a+R) & i(a-R) \\ -1 & i \end{pmatrix}} f_{\begin{pmatrix} -i & i(\bar{a}-R) \\ 1 & -(\bar{a}+R) \end{pmatrix}}(\bar{\alpha})$$

$$= \frac{-(a+R)\left(\dfrac{-i\bar{\alpha} + i(\bar{a}-R)}{\bar{\alpha} - (\bar{a}+R)}\right) + i(a-R)}{\dfrac{i\bar{\alpha} - i(\bar{a}-R)}{\bar{\alpha} - (\bar{a}+R)} + i}$$

$$\begin{pmatrix} -(a+R) & i(a-R) \\ -1 & i \end{pmatrix}\begin{pmatrix} -i & i(\bar{a}-R) \\ 1 & -(\bar{a}+R) \end{pmatrix}$$

$$= \begin{pmatrix} i(a+R)+i(a-R) & -i(a+R)(\bar{a}-R) - i(a-R)(\bar{a}+R) \\ i+i & -i(\bar{a}-R) - i(\bar{a}+R) \end{pmatrix}$$

$$= \begin{pmatrix} 2ia & -i(2|a|^2 + 2R^2) \\ 2i & -2i\bar{a} \end{pmatrix} \rightarrow \frac{a\bar{\alpha} - (|a|^2 + R^2)}{\bar{\alpha} - \bar{a}}$$

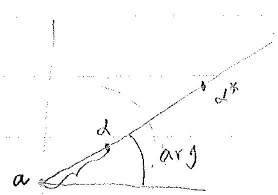
$$= \frac{(a+R)\bar{\alpha} - (a+R)(\bar{a}-R) + (a-R)\bar{\alpha} - (a-R)(\bar{a}+R)}{\bar{\alpha} - (\bar{a}-R) + \bar{\alpha} - (\bar{a}+R)}$$

$$= \frac{2a\bar{\alpha} - |a|^2 + aR - \bar{a}R + R^2 - |a|^2 - aR + \bar{a}R + R^2}{2\bar{\alpha} - 2\bar{a}}$$

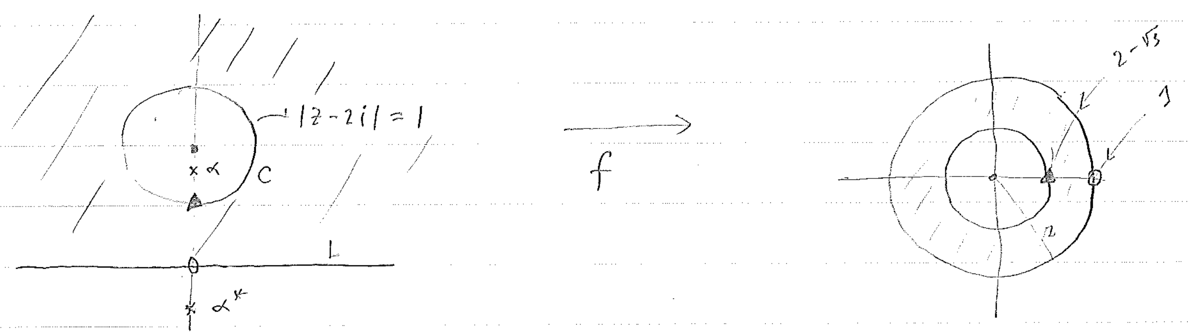
$$= \frac{a\bar{\alpha} - |a|^2 + R^2}{\bar{\alpha} - \bar{a}} = \frac{a(\bar{\alpha} - \bar{a}) + R^2}{\bar{\alpha} - \bar{a}} = \boxed{a + \frac{R^2}{\bar{\alpha} - \bar{a}} = \alpha^*}$$

Interpretation $\quad \alpha^* - a = \dfrac{R^2(\alpha - a)}{|\alpha - a|^2} \Rightarrow \arg(\alpha^* - a) = \arg(\alpha - a)$

$$|\alpha^* - a| = \frac{R^2}{|\alpha - a|}$$



Example: Find a FLT mapping D onto an annulus



Step 1: find α, α^* that are symmetric w.r.t. both C, L.

Step 2: find a FLT mapping $\alpha \to 0$ and $\alpha^* \to \infty$

Then $f(C)$ and $f(L)$ must be symmetric w.r.t. 0 and ∞

i.e. they are circles centered at 0.

Step 1: α^*, α are symm. w.r.t. L if $\alpha^* = \bar{\alpha}$

α^*, α are symm. w.r.t. C $(a = 2i,\ R = 1)$ if $\alpha^* = a + \dfrac{R^2}{\bar{\alpha} - \bar{a}}$

$$= 2i + \frac{1}{\bar{\alpha} + 2i}$$

So solve $\bar{\alpha} = 2i + \dfrac{1}{\bar{\alpha}+2i}$ $\quad (\bar{\alpha} - 2i)(\bar{\alpha} + 2i) = 1 \quad (\bar{\alpha})^2 + 4 = 1$

$\bar{\alpha}^2 = -3 \quad \bar{\alpha} = \pm i\sqrt{3} \qquad \alpha = i\sqrt{3} \quad \alpha^* = -i\sqrt{3}$

Step 2: We want $\alpha \to 0$, $\alpha^* \to \infty$. One more choice say $0 \to 1$

So, $(f(z)) = (z, i\sqrt{3}, 0, -i\sqrt{3})$

$$w = -\frac{z - i\sqrt{3}}{z + i\sqrt{3}} = \frac{-z + i\sqrt{3}}{z + i\sqrt{3}}$$

is the desired mapping

$z = x \Rightarrow |w| = 1$ ✓

$z = 2i + e^{i\theta} \qquad w = -\dfrac{(2-\sqrt{3})i + e^{i\theta}}{(2+\sqrt{3})i + e^{i\theta}}$

$|w| = [(2-\sqrt{3})i + e^{i\theta}][-(2+\sqrt{3})i + e^{-i\theta}]$ hard.

Where does f map L and C?
we know images are circles center at 0, Find radii:

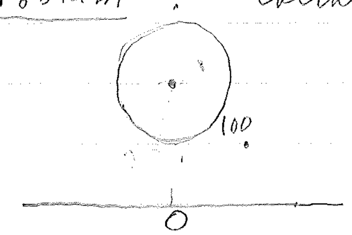
Then $f(L) = \{|w| = 1\}$

$f(C) = ?$ well, $f(i) = -\dfrac{i - i\sqrt{3}}{i + i\sqrt{3}} = \dfrac{\sqrt{3}-1}{\sqrt{3}+1} = \dfrac{(\sqrt{3}-1)^2}{2} = \dfrac{4-2\sqrt{3}}{2} = 2-\sqrt{3}$

So it must be $\{|w| = 2-\sqrt{3}\}$.

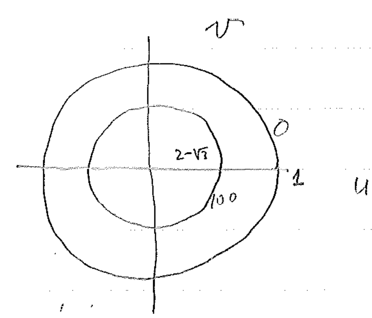
Lecture 22

Problem : Calculate the steady state temperature distn $t(x,y)$ (eg a pipe with boiling water travelling through ground. What is $t(1+i)$?



Need to solve $\Delta t = 0$ with B.C. as shown.

Soln: map via $f(z) = \dfrac{-z + i\sqrt{3}}{z + i\sqrt{3}}$ to



$T(u,v) = A \ln\left(\sqrt{u^2+v^2}\right) + B$

$\left(= \operatorname{Re}(A \log(w) + B)\right)$ ← explain: even though log has a branch cut

$T(2-\sqrt{3}, 0) = 100$ $T(1,0) = 0$

$A \ln(2-\sqrt{3}) + B = 100$ $A \log(1) + B = 0$

$\Rightarrow B = 0 \quad A = \dfrac{100}{\ln(2-\sqrt{3})}$

$T(u,v) = \dfrac{100}{\ln(2-\sqrt{3})} \ln\left(\sqrt{u^2+v^2}\right) = \dfrac{50}{\ln(2-\sqrt{3})} \ln(u^2+v^2)$

$f(z) = \dfrac{-x - iy + i\sqrt{3}}{x + i(y+\sqrt{3})} = \dfrac{(-x + i(-y+\sqrt{3}))(x - i(y+\sqrt{3}))}{x^2 + (y+\sqrt{3})^2}$

$= \underbrace{\dfrac{-x^2 + 3 - y^2}{x^2 + (y+\sqrt{3})^2}}_{u(x,y)} + i \underbrace{\dfrac{\left(x(y+\sqrt{3}) + x(-y+\sqrt{3})\right)}{x^2 + (y+\sqrt{3})^2}}_{v(x,y)}$

$$= \underbrace{\frac{3 - x^2 - y^2}{x^2 + (y+\sqrt{3})^2}}_{u(x,y)} + i \underbrace{\frac{2\sqrt{3}\, x}{x^2 + (y+\sqrt{3})^2}}_{v(x,y)}$$

$x = 0$ $y = \ldots$

$u = 1$ $v = 0$

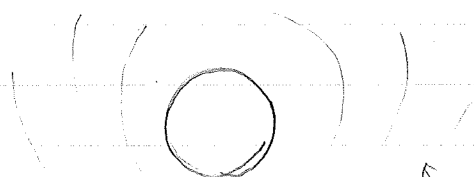
$v = 1$

$$u = \frac{2}{(1+\sqrt{3})^2} = \frac{2}{4+2\sqrt{3}} = \frac{1}{\sqrt{3}+2}$$

$$t(x,y) = T(u(x,y), v(x,y)) = \frac{100}{\ln(2-\sqrt{3})} \frac{1}{2} \ln\left(u^2 + v^2\right)$$

$$= \frac{50}{\ln(2-\sqrt{3})} \ln\left(\frac{x^2 + (y-\sqrt{3})^2}{x^2 + (y+\sqrt{3})^2}\right)$$

$$t(1,1) = \frac{50}{\ln(2-\sqrt{3})} \ln\left(\frac{1 + (1-\sqrt{3})^2}{1 + (1+\sqrt{3})^2}\right) \cong 64.797^\circ$$



← iso therms

$$\frac{\sqrt{3} + y}{\sqrt{3} + y}$$

Temp profile along iy $y \in [0,1]$ $u(0,iy) = \dfrac{3 - y^2}{(y+\sqrt{3})^2}$ $v = 0$

$$t = \frac{100}{\ln(2-\sqrt{3})} \cancel{\tfrac{1}{2}} \ln\left(\frac{\sqrt{3} - y}{\sqrt{3} + y}\right)$$