NOTES ON THE ALGEBRA OF BOUNDED BOREL FUNCTIONS

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Let X be a nonempty set. Recall some usual definitions.

Definition 0.1. A collection $\mathcal{F}$ of subsets of X is called an **algebra** if

(1) $X \in \mathcal{F}$
(2) if $E \in \mathcal{F}$ then $E^c \in \mathcal{F}$
(3) if $E, F \in \mathcal{F}$ then $E \cap F \in \mathcal{F}$ i.e. $\mathcal{F}$ is closed under finite intersections.

Note that we automatically have $\emptyset \in \mathcal{F}$ and that $\mathcal{F}$ is closed under finite unions.

Definition 0.2. the algebra $\mathcal{F}$ is called a **σ-algebra** if in addition we have the axiom

(4) for any sequence $\{E_n\} \subset \mathcal{F}$ we have that $\bigcap_n E_n \in \mathcal{F}$

Note that it automatically follows that $\mathcal{F}$ is also closed under countable unions.

Definition 0.3. A collection $\mathcal{A}$ of functions $X \longrightarrow \mathbb{C}$ is called an **algebra of functions** if

(1) $\mathcal{A}$ is a vector subspace of the vector space of functions $X \longrightarrow \mathbb{C}$
(2) the constant function 1 is in $\mathcal{A}$
(3) if $f, g \in \mathcal{A}$ then $fg \in \mathcal{A}$
(4) if $f \in \mathcal{A}$ then $\overline{f} \in \mathcal{A}$

The following proposition shows a link between the concepts of an algebra and a algebra of functions.

Proposition 0.4. *Let $\mathcal{A}$ be an algebra of functions $X \longrightarrow \mathbb{C}$. Then the collection $\mathcal{F} = \{S \subset X : 1_S \in \mathcal{A}\}$ is an algebra.*

Proof. immediate

(1) $X \in \mathcal{F}$ since $1_X = 1 \in \mathcal{A}$
(2) if $E \in \mathcal{F}$ then $1_{E^c} = 1 - 1_E \in \mathcal{A}$ so that $E^c \in \mathcal{F}$
(3) if $E, F \in \mathcal{F}$ then $1_{E \cap F} = 1_E 1_F \in \mathcal{A}$ hence $E \cap F$ in $\mathcal{F}$

□

Definition 0.5. We say that $\mathcal{A}$ is closed under pointwise limits of uniformly bounded sequences if:

Whenever f_n is a sequence such that

- $\sup_{n,x} |f_n(x)| < \infty$
- $\forall x, \lim_{n \longrightarrow \infty} f_n(x) = f(x)$

then the function $f(x)$ has to be in $\mathcal{A}$.

Proposition 0.6. *Suppose $\mathcal{A}$ is closed under pointwise limits of uniformly bounded sequences. Then $\mathcal{F} = \{S \subset X : 1_S \in \mathcal{A}\}$ is a σ-algebra.*

Proof. By previous proposition we already know that $\mathcal{F}$ is an algebra. Hence (by a standard trick) it suffices to show that whenever $\{E_n\}$ is a decreasing sequence in $\mathcal{F}$, then $E = \bigcap_n E_n \in \mathcal{F}$. Well, 1_{E_n} is uniformly bounded by 1 and $1_E(x) = \lim_{n \longrightarrow \infty} 1_{E_n}(x)$ for all $x \in X$. Hence $1_E \in \mathcal{A}$ and $E \in \mathcal{F}$. □

Now consider the case where X is a metric space. Let $C_b(X)$ be the algebra of bounded continuous functions $X \longrightarrow \mathbb{C}$ and $B_b(X)$ the algebra of bounded Borel functions $X \longrightarrow \mathbb{C}$. Then: $C_b(X) \subset B_b(X)$, $B_b(X)$ is closed under pointwise limits of uniformly bounded sequences but $C_b(X)$ is not.

Lemma 0.7. *$B_b(X)$ is the smallest algebra containing $C_b(X)$ and closed under pointwise limits of uniformly bounded sequences.*

Proof. Let $\mathcal{A}$ be a algebra containing $C_b(X)$ and closed under pointwise limits of uniformly bounded sequences.

Step 1: $\mathcal{A}$ contains characteristic functions of closed sets.

Proof. Let $F \subset X$ be closed. If $F = X$ then $1_F = 1 \in \mathcal{A}$ because $\mathcal{A}$ is an algebra of functions. If F is a proper subset of X we define $F_n = \{x : dist(x, F) \geq 1/n\}$ and the corresponding cut-off functions

$$f_n(x) = \frac{dist(x, F_n)}{dist(x, F_n) + dist(x, F)}$$

Then $f_n \in C_b(X)$, $0 \leq f_n \leq 1$ and for all $x \in X$, $f_n(x) \longrightarrow_n 1_F(x)$. □

Step 2: Since $\mathcal{F} = \{S \subset X : 1_S \in \mathcal{A}\}$ is a σ-algebra containing all closed subsets of X, it contains the Borel σ-algebra of X. Hence $\mathcal{A}$ contains characteristic functions of Borel sets and finite linear combinations of those - simple functions.

Step 3: Any bounded Borel function $h : X \longrightarrow \mathbb{C}$ can be obtained as a pointwise limit of simple functions s_n with $|s_n| \leq |h|$. □

Remark 0.8. One cannot argue that any bounded Borel function is a pointwise limit of a sequence of continuous functions. Indeed suppose that f_n is a sequence of continuous functions converging pointwise to f (WLOG f_n and f are real valued). Then $g_n \downarrow f$ where $g_n = \sup_{k \geq n} f_k$.

Then for any a, the set

$$\{x : f(x) > a\} = \bigcap_n \{x : g_n(x) > a\}$$

is a G_δ (countable intersection of open sets).

Now let $B \subset X$ be a Borel set and $f = 1_B$ its characteristic function. If there exists a sequence of continuous functions f_n converging pointwise to f, it follows that $B = \{x : f(x) > 1/2\}$ is a G_δ. But there are Borel sets are not G_δ's.

1. Uniqueness of functional calculus of self adjoint operators

1.1. **bounded self adjoint operators.** Let $\mathcal{H}$ be a Hilbert space and $\mathcal{B}(\mathcal{H})$ the algebra of bounded operators on $\mathcal{H}$. Let H be a bounded self adjoint operator on $\mathcal{H}$. One first proves (for example in Reed and Simon) the existence and uniqueness of a continuous functional calculus. $\Phi : C(sp(H)) \longrightarrow \mathcal{B}(\mathcal{H})$ s.t.

(1) Φ is an algebraic *-homomorphism
(2) If $f(x) = x$ then $\Phi(f) = H$
(3) $\|\Phi(f)\| = \|f\|_\infty$

Uniqueness is obvious here because the properties (2) and (1) uniquely specify Φ on polynomials and property (3) together with Weierstrass uniquely expends to continuous functions on $sp(H)$.

After that, using spectral measures, one constructs an extension of Φ to a calculus $\widehat{\Phi} : B_b(\mathbb{R}) \longmapsto \mathcal{B}(\mathcal{H})$. s.t.

(1) $\widehat{\Phi}$ is an algebraic *-homomorphism
(2) If $f(x) = x$ then $\widehat{\Phi}(f) = H$
(3) $\left\|\widehat{\Phi}(f)\right\| \leq \|f\|_\infty$
(4) If $f_n(x)$ is a uniformly bounded sequence converging pointwise to $f(x)$ then $\widehat{\Phi}(f_n)$ converges strongly to $\widehat{\Phi}(f)$

Uniqueness is a consequence of lemma 0.7 by the following argument. Suppose we have two extensions $\widehat{\Phi}$ and $\widehat{\Phi}'$. Let

$$\mathcal{A} = \left\{f \in B_b(\mathbb{R}) : \widehat{\Phi}(f) = \widehat{\Phi}'(f)\right\}.$$

Then $\mathcal{A}$ is an algebra of function containing $C_b(\mathbb{R})$. Property (4) shows that $\mathcal{A}$ is closed under pointwise limits of uniformly bounded sequences. Hence $\mathcal{A} = B_b(\mathbb{R})$.

1.2. **unbounded self adjoint operators (the main idea).** Uniqueness of the calculus is clear on resolvent functions $r_z(x) = 1/(x-z)$ and finite linear combinations of those. Then Weirstrass theorem allows to extend the calculus uniquely to $C_0(\mathbb{R}) = \left\{f \in C(\mathbb{R}) : \lim_{|x| \longrightarrow \infty} f(x) = 0\right\}$. After that extend the calculus to $C_b(\mathbb{R})$ by approximating $f(x) = \lim_{n \longrightarrow \infty} 1_{|x| \leq n} f(x)$ and using property (4). The extension to $B_b(\mathbb{R})$ from $C_b(\mathbb{R})$ uses exactly the same argument as for bounded operators given above.

2. References

Reed and Simon vol 1. PB 16 chapter 1.