Math 257 and 316

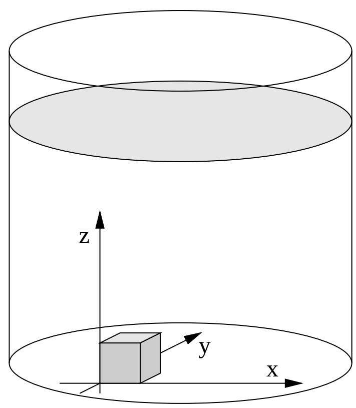
Partial Differential Equations

Introduction

This is a course about partial differential equations, or PDE's. These are differential equations involving partial derivatives and multi-variable functions.

An example: temperature flow

Consider the following experiment. A copper cube (with side length $10\mathrm{cm}$) is taken from a refrigerator (at temperature -4°) and, at time $t = 0$ is placed in a large pot of boiling water. What is the temperature of the centre of the cube after 60 seconds?



Introduce the x, y and z co-ordinate axes, with the origin located at one corner of the cube as shown. Let $T(x, y, z, t)$ be the temperature of the point (x, y, z) inside the cube at time t. Then $T(x, y, z, t)$ is defined for all (x, y, z) inside the cube (i.e., $0 \le x, y, z \le 10$), and all $t \ge 0$. Our goal is to determine the function T, in particular, to find $T(5, 5, 5, 100)$.

There are three facts about $T(x, y, z, t)$ that are needed to determine it completely. Firstly, T solves the partial differential equation governing heat flow, called the heat equation. This means that for every (x, y, z) in the cube and every $t > 0$,

$$\frac{\partial T}{\partial t} = \alpha^2 \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right)$$

Here α^2 is a constant called the thermal diffusivity. This constant depends on what kind of metal the cube is made of. For copper $\alpha^2 = 1.14\,\mathrm{cm}^2/\mathrm{sec}$

The second fact about T is that it satisfies a boundary condition. We assume that the pot of water is so large that the temperature of the water is not affected by the insertion of the cube. This means that the boundary of the cube is always held at the constant temperature 100°. In other words

$$T(x, y, z, t) = 100 \quad \text{for all } (x, y, z) \text{ on the boundary of the cube and all } t > 0.$$

The third fact about T is that it satisfies an initial condition. At time $t = 0$ the cube is at a uniform temperature of -4°. This means that

$$T(x, y, z, 0) = -4 \quad \text{for all } (x, y, z) \text{ inside the cube.}$$

Given these three facts we can determine T completely. In this course we will learn how to write down the solution in the form of an infinite (Fourier) series. For now I'll just write down the answer.

$$T(x,y,z,t) = 100 + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{l=1}^{\infty} a_{n,m,l} e^{-\alpha^2 \lambda_{m,n,l} t} \sin(n\pi x/10)\sin(m\pi y/10)\sin(l\pi z/10)$$

where

$$\lambda_{m,n,l} = (\pi/10)^2(n^2 + m^2 + l^2)$$

and

$$a_{n,m,l} = -104\left(\frac{2}{\pi}\right)^3 \frac{1}{nml}(1-(-1)^n)(1-(-1)^m)(1-(-1)^l)$$

Problem 1.1: What temperature does the formula above predict for the centre of the cube at time $t = 60$ if you use just one term ($(n,m,l) = (1,1,1)$) in the sum. What is the answer if you include the terms $(n,m,l) = (1,1,1), (3,1,1), (1,3,1), (1,1,3)$?

Basic equations

The three basic equations that we will consider in this course are the heat equation, the wave equation and Laplace's equation. We have already seen the heat equation in the section above. The sum of the second partial derivatives with repsect to the space variables (x, y and z) is given a special name—the Laplacian or the Laplace operator—and is denoted Δ. Thus the Laplacian of u is denoted

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

With this notation the heat equation can be written

$$\frac{\partial u}{\partial t} = \alpha^2 \Delta u \tag{1.1}$$

Sometimes we are interested in situations where there are fewer than three space variables. For example, if we want to describe heat flow in a thin plate that is insulated on the top and the bottom, then the temperature function only depends on two space variables, x and y. The heat flow in a thin insulated rod would be described by a function depending on only one space variable x. In these situations we redefine Δ to mean

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

or

$$\Delta u = \frac{\partial^2 u}{\partial x^2}$$

depending on the context. The heat equation is then still written in the form (1.1).

The second basic equation is the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u$$

This equation decribes a vibrating string (one space dimension) a vibrating drum or water waves (two space dimensions) or an electric field propagating in free space (three space dimensions).

The third basic equation is the Laplace equation

$$\Delta u = 0$$

Notice that a solution of Laplace's equation can be considered to be a solution of the wave or heat equation that is constant in time (so that $\partial u/\partial t = \partial^2 u/\partial t^2 = 0$). So we see that solutions to Laplace's equation describe equilibrium situations. For example, the temperature distrubution after a long time a passed, or the electric field produced by stationary charges are solutions of Laplace's equation.

These are all second order equations because there are at most two derivatives applied to the unknown function.

These equations are also all linear. This means that if $u_1(x,y,z,t)$ and $u_2(x,y,z,t)$ are solutions then so is the linear combination $a_1u_1(x,y,z,t)+a_2u_2(x,y,z,t)$. This fact is sometimes called the superpostion principle. For example, if u_1 and u_2 both satisfy the heat equation, then

$$\begin{aligned}\frac{\partial}{\partial t}(a_1u_1+a_2u_2) &= a_1\frac{\partial u_1}{\partial t}+a_2\frac{\partial u_2}{\partial t}\\ &= a_1\alpha^2\Delta u_1 + a_2\alpha^2\Delta u_2\\ &= \alpha^2\Delta(a_1u_1+a_2u_2)\end{aligned}$$

Thus the linear combination $a_1u_1+a_2u_2$ solves the heat equation too. This reasoning can be extended to a linear combination of any number, even infinitely many, solutions.

Here is how the superposition principle is applied in practice. We try to get a good supply of basic solutions that satisfy the equation, but not neccesarily the desired boundary condition or the initial condition. These basic solutions are obtained by a technique called separation of variables. Then we try to form a (possibly infinite) linear combination of the basic solutions in such a way that the initial condition and boundary condition are satisfied as well.

Problem 1.2: Show that for each fixed n, m and l the function

$$e^{-\alpha^2\lambda_{m,n,l}t}\sin(n\pi x/10)\sin(m\pi x/10)\sin(l\pi x/10)$$

is a solution of the heat equation. What is the boundary condition satisfied by these functions? Is this boundary condition preserved when we form a linear combination of these functions? Notice that the solution T above is an infinite linear combination of these functions, together with constant function 100.

Problem 1.3: What is the limiting temperature distribution in the example as $t\to\infty$? Show that is an (admitedly pretty trivial) solution of the Laplace equation.

Problem 1.4: When there is only one space variable, then Laplace's equation is the ordinary differential equation $u''=0$ rather than a partial differential equation. Write down all possible solutions.

If you have taken a course in complex analysis, you will know that the real part (and the imaginary part) of every analytic function is a solution to Laplace's equation in two variables. Thus there are many more solutions to Laplace's equations when it is a PDE.

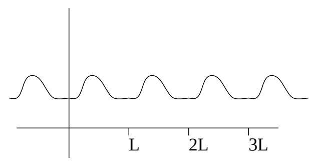
What else is in this course?

The series arising in the solutions of the basic equations are often Fourier series. Fourier series are of independent interest, and we will devote some time at the beginning of this course studying them.

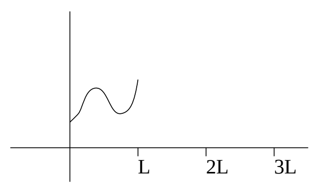
Fourier series expansions can be considered as special cases of expansions in eigenfunctions of Sturm-Liouville problems. More general Sturm-Liouville expansions arise when considering slighly more complicated (and realistic) versions of the basic equations, or when using non-Cartesian co-ordinate systems, such as polar co-ordinates. We will therefore spend some time looking at Sturm-Liouville problems.

Fourier series

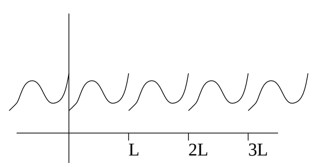
Consider a function $f(x)$ of one variable that is periodic with period L. This means that $f(x+L) = f(x)$ for every x. Here is a picture of such a function.



Such a function is completely determined by its values on any interval of length L, for example $[0, L]$ or $[-L/2, L/2]$. If we start with any function defined on an interval of length L, say $[0, L]$,



we can extend it to be a periodic function by simply repeating its values.



Notice, however, that the periodic extention will most likely have a jump at the points where it is glued together, and not be a continuous function.

The Fourier series for f is an infinite series expansion of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(2\pi nx/L) + b_n \sin(2\pi nx/L)$$

Each function appearing on the right side—the constant function 1, the functions $\cos(2\pi nx/L)$ and the functions $\sin(2\pi nx/L)$ for $n = 1, 2, 3, \ldots$—are all periodic with period L. Apart form this, though, there is not any apparent reason why such an expansion should be possible.

For the moment, we will just assume that it is possible to make such an expansion, and try to determine what the coefficients a_n and b_n must be. The basis for this determination are the following integral formulas, called orthogonality relations.

$$\int_0^L \cos(2\pi nx/L)dx = \int_0^L \sin(2\pi nx/L)dx = 0 \quad \text{for every } n \tag{2.2}$$

$$\int_0^L \cos(2\pi nx/L)\sin(2\pi mx/L)dx = 0 \quad \text{for every } n \text{ and } m \tag{2.3}$$

$$\int_0^L \sin(2\pi nx/L)\sin(2\pi mx/L)dx = \begin{cases} L/2 & \text{if } n = m \\ 0 & \text{otherwise} \end{cases} \tag{2.4}$$

$$\int_0^L \cos(2\pi nx/L)\cos(2\pi mx/L)dx = \begin{cases} L/2 & \text{if } n = m \\ 0 & \text{otherwise} \end{cases} \tag{2.5}$$

Notice that we can change $\int_0^L$ to $\int_{-L/2}^{L/2}$ (or to the integral over any other interval of length L) without changing the values of the integrals. This can be seen geometrically: the area under the curve of a periodic function between $x = L/2$ and $x = L$ is the same as the area under the curve between $-L/2$ and $x = 0$, so we can shift that part of the integral over.

If we want to determine a_m (for $m \geq 1$) we first multiply both sides of (2.1) by $\cos(2\pi mx/L)$ and integrate. This gives

$$\int_0^L \cos(2\pi mx/L)f(x)dx = \frac{a_0}{2}\int_0^L \cos(2\pi mx/L)dx +$$
$$\int_0^L \left(\sum_{n=1}^{\infty} a_n \cos(2\pi mx/L)\cos(2\pi nx/L) + b_n \cos(2\pi mx/L)\sin(2\pi nx/L)\right)dx$$

Now we change the order of summation and integration. This is not always justfied (i.e., summing first and then integrating could give a different answer than integrating first and then summing). However, at the moment, we don't really even know that the Fourier series converges, and are just trying to come up with some formula for a_m. So we will go ahead and make the exchange, obtaining

$$\int_0^L \cos(2\pi mx/L)f(x)dx = \frac{a_0}{2}\int_0^L \cos(2\pi mx/L)dx +$$
$$\sum_{n=1}^{\infty} a_n \int_0^L \cos(2\pi mx/L)\cos(2\pi nx/L)dx + b_n \int_0^L \cos(2\pi mx/L)\sin(2\pi nx/L)dx$$

Using the orthogonality realtions, we see that all the terms are zero except the one with $n = m$. Thus

$$\int_0^L \cos(2\pi mx/L)f(x)dx = a_m \int_0^L \cos^2(2\pi mx/L)dx = a_m L/2$$

so that

$$a_m = \frac{2}{L}\int_0^L \cos(2\pi mx/L)f(x)dx.$$

Similary, multiplying (2.1) by $\sin(2\pi mx/L)$, integrating, changing the order of summation and integration, and using the orthogonality relations, we obtain

$$b_m = \frac{2}{L}\int_0^L \sin(2\pi mx/L)f(x)dx.$$

Finally, if we multiply (2.1) by 1 (which does nothing, of course) and follow the same steps we get

$$a_0 = \frac{2}{L}\int_0^L f(x)dx$$

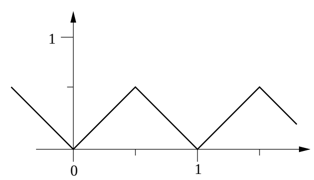
(This can also be written $2/L\int_0^L \cos(2\pi 0x/L)f(x)dx$, and agrees with the formula for the other a_m's. This is the reason for the factor of $1/2$ in the $a_0/2$ term in the original expression for the Fourier series.)

Examples

Given a periodic function f (or a function on $[0, L]$ which we extend periodically) we now have formulas for the coefficients a_n and b_n in the fourier expansion. We can therefore compute all the terms in the fourier expansion, and see if it approximates the original function.

Lets try this for a triangular wave defined on $[0, 1]$ by

$$f(x) = \begin{cases} x & \text{if } 0 \le x \le 1/2 \\ 1-x & \text{if } 1/2 \le x \le 1 \end{cases}$$



In this example $L = 1$. Using the formulas from the last section we have

$$a_0 = 2\int_0^1 f(x)dx = 1/2$$

(since the area under the triangle is $1/4$). For $n \ge 1$ we obtain (after some integration by parts)

$$\begin{aligned} a_n &= 2\int_0^1 \cos(2\pi nx)f(x)dx \\ &= 2\int_0^{1/2} \cos(2\pi nx)xdx + 2\int_{1/2}^1 \cos(2\pi nx)(1-x)dx \\ &= ((-1)^n - 1)/(\pi^2 n^2) \\ &= \begin{cases} 0 & \text{if } n \text{ is even} \\ -2/(\pi^2 n^2) & \text{if } n \text{ is odd} \end{cases} \end{aligned}$$

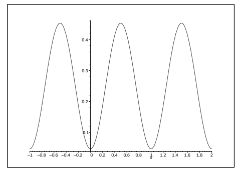
and

$$\begin{aligned} b_n &= 2\int_0^1 \sin(2\pi nx)f(x)dx \\ &= 2\int_0^{1/2} \sin(2\pi nx)xdx + 2\int_{1/2}^1 \sin(2\pi nx)(1-x)dx \\ &= 0 \end{aligned}$$

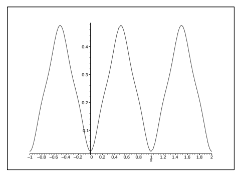
Thus, if the Fourier series for $f(x)$ really does converge to $f(x)$ we have,

$$\begin{aligned} f(x) &= \frac{1}{4} - 2\sum_{\substack{n=1 \\ n\text{odd}}}^{\infty} \frac{1}{\pi^2 n^2}\cos(2\pi nx) \\ &= \frac{1}{4} - 2\sum_{n=0}^{\infty} \frac{1}{\pi^2(2n+1)^2}\cos(2\pi(2n+1)x) \end{aligned}$$

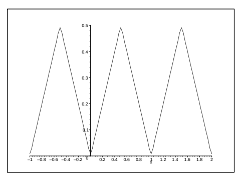
Does the series converge to f? We can try to look at the partial sums and see how good the approximation is. Here is a graph of the first 2, 3 and 7 non-zero terms.



2 non zero terms



3 non zero terms

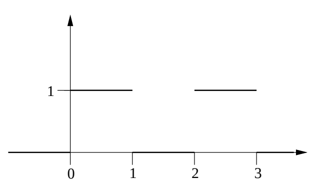


7 non zero terms

It looks like the series is converging very quickly indeed.

Lets try another example. This time we take $L = 2$ and $f(x)$ to be a square wave given by

$$f(x) = \begin{cases} 1 & \text{if } 0 \le x < 1 \\ 0 & \text{if } 1 \le x \le 2 \end{cases}$$



Notice that this function is not continous at $x = 1$, and since we should be thinking of it as a periodic function with period 2, it is also not continuous at $x = 0$ and $x = 2$. Using the formulas from the previous section we find

$$a_0 = 1$$

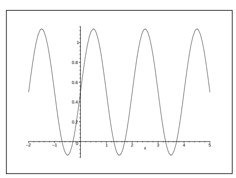
$$
\begin{aligned}
a_n &= \frac{2}{2}\int_0^2 \cos((2\pi n x)/2) f(x)dx \\
&= \int_0^1 \cos(\pi n x)dx \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
b_n &= \frac{2}{2}\int_0^2 \sin((2\pi n x)/2) f(x)dx \\
&= \int_0^1 \sin(\pi n x)dx \\
&= (1-(-1)^n)/(\pi n) \\
&= \begin{cases} 0 & \text{if } n \text{ is even} \\ 2/(\pi n) & \text{if } n \text{ is odd} \end{cases}
\end{aligned}
$$

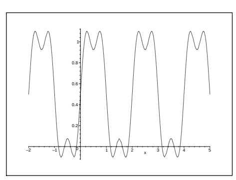
Thus, if the series for this f exists, it is given by

$$
\begin{aligned}
f(x) &= \frac{1}{2} + \sum_{\substack{n=1 \\ n \text{odd}}}^{\infty} \frac{2}{\pi n}\sin(\pi n x) \\
&= \frac{1}{2} + \sum_{n=0}^{\infty} \frac{2}{\pi(2n+1)}\sin(\pi(2n+1)x)
\end{aligned}
$$

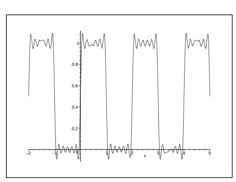
Here are the graphs of the first 2, 3, 7 and 20 non-zero terms in the series.



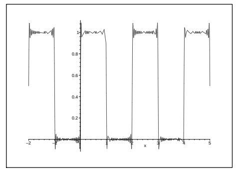
2 non zero terms



3 non zero terms



7 non zero terms



20 non zero terms

The series seems to be converging, although the convergence doesn't seem so good near the discontinuities. The fact that there is a bump near the discontinuity is called Gibb's phenonemon. The bump moves closer and closer to the point of discontinuity as more and more terms in the series are taken. So for any fixed x, the bump eventually passes by. But it never quite goes away. (For those of you who know about uniform convergence: this is an example of a series that converges pointwise but not uniformly.)

Problem 2.1: Compute the coefficents a_n and b_n when $L = 1$ and

$$f(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1/2 \\ -1 & \text{if } 1/2 < x \le 1 \end{cases}$$

Problem 2.2: Compute the coefficents a_n and b_n when $L = 2$ and

$$f(x) = x \quad \text{if } 0 \le x \le 2$$

The Fourier theorem

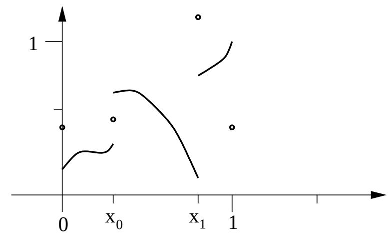
Suppose that $f(x)$ is a function for which the integrals defining the coefficients a_n and b_n exist. This would be true, for example, if f is continuous. Then it is possible to define the Fourier series for f. Does this Fourier series converge to f?

It turns out the answer is: not always! The existence of Gibbs phenonmenon might make you suspect that things are not completely simple. And in fact, the complete answer is very complicated. However, for most functions encountered in practice, the Fourier series does converge. Here is a theorem.

Theorem 2.1 *Suppose that $f(x)$ is a periodic function with period L, and that both f and its derivative f' are continuous. Then the Fourier series for f converges to f.*

This theorem is nice, but doesn't cover either of the examples above. For the triangle wave, f is continuous, but f' is not. And for the square wave neither f nor f' is continuous.

Definition: A function $f(x)$ is called piecewise continuous if there are finitely many points $x_0, x_1, \ldots, x_k$ such that f is continous on each interval between the points x_i and f has finite limits from the left and the right (which need not be equal) at each of the points x_i. Here is a typical piecewise continuous function. The value of the function right at the points x_i is irrelevant.



Theorem 2.2 *Suppose that $f(x)$ is a periodic function with period L, and that both f and its derivative f' are piecewise continuous on each interval of length L. Then at all points of continuity x the Fourier series evaluated at x converges to $f(x)$. At the finitely many points of discontinuity x_k, the Fourier series evaluated at x_k converges to the average of the limits from the right and the left of f, i.e., to $(f(x_k+)+f(x_k-))/2$.*

Although we won't be able to prove this theorem in this course, here is some idea how one would go about it. Let $F_n(x)$ be the partial sum

$$F_m(x) = \frac{a_0}{2} + \sum_{n=1}^{m} a_n \cos(2\pi nx/L) + b_n \sin(2\pi nx/L)$$

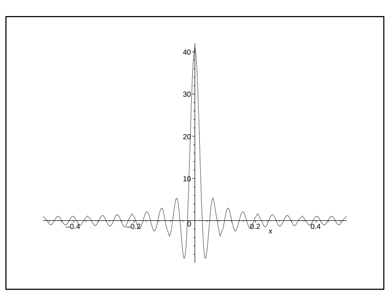
Our goal would be to show that $F_m(x) \to f(x)$ as $m \to \infty$. If we substitute the formulas for a_n and b_n we get

$$\begin{aligned}
F_m(x) &= \frac{1}{2}\frac{2}{L}\int_0^L f(y)dy \\
&\quad + \sum_{n=1}^{m} \left(\frac{2}{L}\int_0^L \cos(2\pi my/L)f(y)dy \right) \cos(2\pi nx/L) + \left(\frac{2}{L}\int_0^L \sin(2\pi my/L)f(y)dy \right) \sin(2\pi nx/L) \\
&= \int_0^L \left(\frac{1}{L} + \frac{2}{L}\sum_{n=1}^{m} \cos(2\pi mx/L)\cos(2\pi ny/L) + \sin(2\pi mx/L)\sin(2\pi ny/L) \right) f(y)dy \\
&= \int_0^L \left(\frac{1}{L} + \frac{2}{L}\sum_{n=1}^{m} \cos(2\pi m(x-y)/L) \right) f(y)dy \\
&= \int_0^L K_m(x-y)f(y)dy
\end{aligned}$$

where

$$K_m(x-y) = \frac{1}{L} + \frac{2}{L}\sum_{n=1}^{m} \cos(2\pi m(x-y)/L)$$

to understand why $F_m(x)$ should converge to $f(x)$ we have to examine the function $K_m(x-y)$. Here is a picture (with $m = 20$, $x = 0$ and $L = 1$).



Notice that there is a big spike when y is close to x. The area under this spike is approximately 1. For values of y away from x, the function is oscillating. When m gets large, the spike get concentrated closer and closer to x and the the oscillations get more and more wild. So

$$\int_0^L K_m(x-y)f(y)dy = \int_{y \text{ close to } x} K_m(x-y)f(y)dy + \int_{y \text{ far from } x} K_m(x-y)f(y)dy$$

When m is very large we can restrict the first integral over y values that are so close to x that $f(y)$ is essentially equal to $f(x)$. Then we have

$$\int_{y \text{ close to } x} K_m(x-y)f(y)dy \sim \int_{y \text{ close to } x} K_m(x-y)f(x)dy = f(x)\int_{y \text{ close to } x} K_m(x-y)dy \sim f(x)$$

because the area under the spike is about 1. On the other hand

$$\int_{y \text{ far from } x} K_m(x-y)f(y)dy \sim 0$$

for large m since the wild oscillations tend to cancel out in the integral. To make these ideas exact, one needs to assume more about the function f, for example, the assumptions made in the theorems above.

Complex form

We will now the Fourier series in complex exponential form.

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i2\pi nx}. \tag{2.6}$$

(To simplify the formulas, we will assume that $L = 1$ in this section.) Recall that

$$e^{it} = \cos(t) + i\sin(t).$$

Therefore

$$\begin{aligned}\cos(t) &= \frac{e^{it}+e^{-it}}{2}\\ \sin(t) &= \frac{e^{it}-e^{-it}}{2i}.\end{aligned}$$

To obtain the complex exponential form of the fourier series we simply substitute these expressions into the original series. This gives

$$\begin{aligned}f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(\frac{a_n}{2}\left(e^{i2\pi nx}+e^{-i2\pi nx}\right) + \frac{b_n}{2i}\left(e^{i2\pi nx}-e^{-i2\pi nx}\right)\right)\\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(\left(\frac{a_n}{2}+\frac{b_n}{2i}\right)e^{i2\pi nx} + \left(\frac{a_n}{2}-\frac{b_n}{2i}\right)e^{-i2\pi nx}\right)\\ &= \sum_{n=-\infty}^{\infty} c_n e^{i2\pi nx}\end{aligned}$$

where

$$\begin{aligned}c_0 &= \frac{a_0}{2}\\ c_n &= \frac{a_n}{2}+\frac{b_n}{2i} \quad \text{for} \quad n>0\\ c_n &= \frac{a_{-n}}{2}-\frac{b_{-n}}{2i} \quad \text{for} \quad n<0.\end{aligned}$$

This complex form of the Fourier series is completely equivalent to the original series. Given the a_n's and b_n's we can compute the c_n's using the formula above, and conversely, given the c_n's we can solve for

$$\begin{aligned} a_0 &= 2c_0 \\ a_n &= c_n + c_{-n} \quad \text{for} \quad n > 0 \\ b_n &= ic_n - ic_{-n} \quad \text{for} \quad n > 0 \end{aligned}$$

It is actually often easier to compute the c_n's directly. To do this we need the appropriate orthogonality relations for the functions $e^{i2\pi nx}$. They are

$$\int_0^1 e^{-i2\pi mx} e^{i2\pi nx} dx = \int_0^1 e^{i2\pi(n-m)x} dx = \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m \end{cases}$$

So to pick out the coefficient c_m in the complex Fourier series, we multiply (2.6) by $e^{-i2\pi mx}$ and integrate. This gives (after exchanging the integral and the infinite sum)

$$\int_0^1 e^{-i2\pi mx} f(x) dx = \sum_{n=-\infty}^{\infty} c_n \int_0^1 e^{-i2\pi mx} e^{i2\pi nx} dx = c_m$$

Lets compute the complex Fourier series coefficients for the square wave function

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1/2 \\ -1 & \text{if } 1/2 < x \leq 1 \end{cases}$$

If $n = 0$ then $e^{-i2\pi nx} = e^0 = 1$ so c_0 is simply the integral of f.

$$c_0 = \int_0^1 f(x) dx = \int_0^{1/2} 1 dx - \int_{1/2}^1 1 dx = 0$$

Otherwise, we have

$$\begin{aligned} c_n &= \int_0^1 e^{-i2\pi nx} f(x) dx \\ &= \int_0^{1/2} e^{-i2\pi nx} dx - \int_{1/2}^1 e^{-i2\pi nx} dx \\ &= \left. \frac{e^{-i2\pi nx}}{-i2\pi n} \right|_{x=0}^{x=1/2} - \left. \frac{e^{-i2\pi nx}}{-i2\pi n} \right|_{x=1/2}^{x=1} \\ &= \frac{2 - 2e^{i\pi n}}{2\pi i n} \\ &= \begin{cases} 0 & \text{if} \quad n \text{ is even} \\ 2/i\pi n & \text{if} \quad n \text{ is odd} \end{cases} \end{aligned}$$

Thus we conclude that

$$f(x) = \sum_{\substack{n=-\infty \\ n \text{ odd}}}^{\infty} \frac{2}{i\pi n} e^{i2\pi nx}$$

We can deduce the original form of the Fourier series from this. Using $a_n = c_n + c_{-n}$ and $b_n = ic_n - ic_{-n}$ we find that $a_n = 0$ for all n, $b_n = 0$ for n even and $b_n = 4/\pi n$ for n odd. Thus

$$f(x) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n} \sin(2\pi nx) = \sum_{n=0}^{\infty} \frac{4}{\pi(2n+1)} \sin(2\pi(2n+1)x)$$

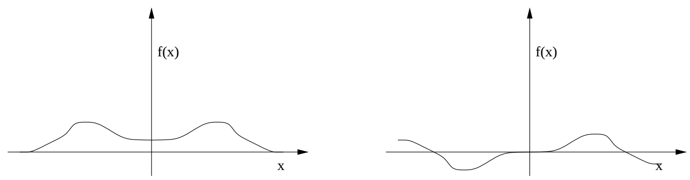
Problem 2.3: Calculate the formula for the c_n's when L is different from 1. Use your formula to compute the coefficents c_n when $L = 2$ and

$$f(x) = x \quad \text{if } 0 \le x \le 2$$

Calculate a_n and b_n from your expression for the c_n and compare to the result obtained in a previous problem.

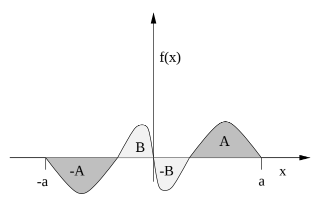
Even and Odd functions

A function $f(x)$ is called even if $f(-x) = f(x)$ and odd if $f(x) = -f(-x)$ Here is are picture of an even function and an odd function.



If we multiply together two even functions or two odd functions, the result is an even function. If we multiply together and odd and an even function, the result is an odd function.

The integral of an odd function over an interval that is symmetric with respect to the origin is zero. This can be seen geometrically:



The integral from $-a$ to a of this odd function is zero, since the positive areas on one side of the origin cancel the negative ones on the other side. Similarly, the integral from $-a$ to a of an even function is just twice the integral of the same function. Thus

$$\int_{-a}^{a} \text{odd}(x)dx = 0 \qquad \int_{-a}^{a} \text{even}(x)dx = 2\int_{0}^{a} \text{even}(x)dx$$

These ideas can be applied to the calculation of Fourier coefficients because $\cos(2\pi nx/L)$ is an even function (for every n) and $\sin(2\pi nx/L)$ is an odd function (for every n). Recall that the interval of integration appearing in the definition of a_n and $b_n$can be any interval of length L. Before we chose $[0, L]$. But now, to apply the formulas for odd and even functions we want an interval that is symmetric about zero. So we choose $[-L/2, L/2]$ and write

$$a_n = \frac{2}{L}\int_{-L/2}^{L/2} \cos(2\pi nx/L)f(x)dx,$$

$$b_n = \frac{2}{L}\int_{-L/2}^{L/2} \sin(2\pi nx/L)f(x)dx.$$

If $f(x)$ is even, then $\cos(2\pi nx/L)f(x)$ is also even and $\sin(2\pi nx/L)f(x)$ is odd. Thus

$$\begin{aligned} a_n &= \frac{4}{L}\int_0^{L/2} \cos(2\pi nx/L)f(x)dx, \\ b_n &= 0. \end{aligned}$$

If $f(x)$ is odd, then $\cos(2\pi nx/L)f(x)$ is odd and $\sin(2\pi nx/L)f(x)$ is even. Thus

$$\begin{aligned} a_n &= 0, \\ b_n &= \end{aligned}$$

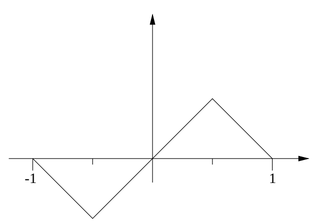
$$\frac{4}{L}\int_0^{L/2} \sin(2\pi nx/L)f(x)dx.$$

Example

Let us compute the Fourier coeffients of the function $f(x)$ with period $L = 2$ given by

$$f(x) = \begin{cases} -1-x & \text{if } -1 \le x \le -1/2 \\ x & \text{if } -1/2 \le x \le 1/2 \\ 1-x & \text{if } 1/2 \le x \le 1 \end{cases}$$

Here is a picture of f.



Notice that f is odd. Thus $a_n = 0$ for every n and

$$\begin{aligned} b_n &= \frac{4}{2}\int_0^1 \sin(2\pi nx/2)f(x)dx \\ &= 2\int_0^{1/2} \sin(\pi nx)xdx + 2\int_{1/2}^1 \sin(\pi nx)(1-x)dx \\ &= 4\sin(n\pi/2)/(\pi^2 n^2) \end{aligned}$$

Problem 2.4: Find the Fourier coefficients of the function with period $L = 4$ given by

$$f(x) = \begin{cases} 0 & \text{if } -2 \le x < -1 \\ 2 & \text{if } -1 \le x < 1 \\ 0 & \text{if } 1 \le x \le 2 \end{cases}$$

Problem 2.5: Let $f_1(x)$ and $f_2(x)$ be two even functions and $g_1(x)$ and $g_2(x)$ be two odd functions. Show that $f_1(x)f_2(x)$ and $g_1(x)g_2(x)$ are even and that $f_1(x)g_1(x)$ is odd.

Problem 2.6: Suppose f and g are odd and even functions that have been shifted vertically, i.e., $f(x) = C + \mathrm{odd}(x)$ and $g(x) = C + \mathrm{even}(x)$, where C is a constant. Which Fourier coefficients are zero?

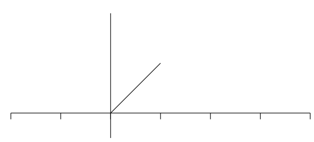
Problem 2.7: Suppose that $L = 1$ and $f(x)$ is odd with respect to $1/2$, i.e., $f(1/2 + x) = f(1/2 - x)$ for every x. What can you say about the Fourier coefficients of f?

Sine and Cosine series

Suppose we are given two different periodic functions, with possibly different periods. Suppose, though, that these two functions happen to agree on some interval. Then the Fourier series for the two functions will be different. In fact, if the periods are different then there will be completely different set of sine and cosine functions appearing in the two expansions. Nevertheless, in the interval where the two functions agree, the two Fourier expansions will converge to the same thing.

We will use this observation, together with the facts about odd and even functions, to produce a variety of expansions for the same function.

We begin with a function defined on the interval $[0, 1]$. For definiteness, take $f(x) = x$.



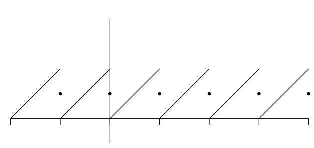
We can extend this function to be periodic with period 1 and expand in a Fourier series. The formulas for the coefficients are

$$a_n = \frac{2}{1}\int_0^1 \cos(2\pi n x) x dx = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n > 1 \end{cases}$$
$$b_n = \frac{2}{1}\int_0^1 \sin(2\pi n x) x dx = -1/(\pi n)$$

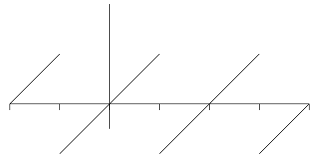
and the series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(2\pi n x) + b_n \sin(2\pi n x) = \frac{1}{2} - \sum_{n=1}^{\infty} \frac{1}{\pi n}\sin(2\pi n x) \tag{2.7}$$

converges to a periodic function of period 1 that agrees with $f(x) = x$ on $[0, 1]$ (except right at the discontinuity, where the Fourier series converges to $1/2$). Here is a picture of the Fourier series.



Next, we will start with the same function $f(x) = x$ on the interval $[0, 1]$, but first extend it to be an odd function on $[-1, 1]$, and then extend it to be a periodic function of period $L = 2$. The complete extension looks like



What is the Fourier series for this function. Well, now $L = 2$ and the function is odd. So all the a_n's are zero, and

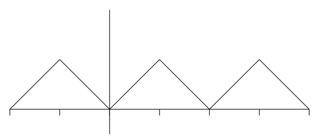
$$b_n = \frac{4}{2}\int_0^1 \sin(n\pi x)x dx = 2(-1)^{n+1}/(\pi n)$$

So the series is

$$\sum_{n=1}^{\infty} b_n \sin(\pi n x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{\pi n}\sin(n\pi x) \tag{2.8}$$

This is called the sine series for f. It converges to the periodic function with period $L = 2$ depicted above (except for the points of discontinuity, where it converges to the midpoint).

Finally, we start again with the same function $f(x) = x$ on the interval $[0, 1]$, extend it to be an even function on $[-1, 1]$, and then extend it to be a periodic function of period $L = 2$. The complete extension now looks like



This is an even function with $L = 2$. So all the b_n's are now zero, while the a_n's are given by

$$\frac{4}{2}\int_0^1 \cos(\pi n x)x dx = \begin{cases} 1 & \text{if } n = 0 \\ 2((-1)^n - 1)/(\pi^2 n^2) & \text{if } n \geq 1 \end{cases}$$

Therefore the series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\pi n x) = \frac{1}{2} + \sum_{n=1}^{\infty} 2\frac{(-1)^n - 1}{\pi^2 n^2}\cos(\pi n x) \tag{2.9}$$

converges to the periodic function depicted above. This is called the cosine series.

If we restrict our attention to values of x in the interval $[0, 1]$ we have produced three different expansions for the same function, namely (2.7), (2.8) and (2.9). (One could produce even more, by making different extensions.) We will see that all these series are useful in solving PDE's.

Problem 2.8: Which of the expansions (2.7), (2.8), (2.9) converges the fastest? Which of the resulting periodic functions are continuous?

Problem 2.9: Compute the sine and cosine series for

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1/2 \\ -1 & \text{if } 1/2 < x \leq 1 \end{cases}$$

Problem 2.10: Suppose $f(x)$ is a function defined on $[0, a]$ (rather than $[0, 1]$). What are the sine and cosine series for this function. Write down the formulas for the coefficients.

Infinite orthogonal bases

Lets write down functions appearing on the right side of the expansions of the previous section. We have

$$\{\frac{1}{2}, \cos(2\pi x), \cos(4\pi x), \cos(6\pi x), \ldots \sin(2\pi x), \sin(4\pi x), \sin(6\pi x), \ldots\}$$

for (2.7),

$$\{\sin(\pi x), \sin(2\pi x), \sin(3\pi x), \ldots\}$$

for (2.8) and

$$\{\frac{1}{2}, \cos(\pi x), \cos(2\pi x), \cos(3\pi x), \ldots\}$$

for (2.9). We could also add the set

$$\{\ldots, e^{-4i\pi x}, e^{-2i\pi x}, e^{0i\pi x}, e^{2i\pi x}, e^{4i\pi x}, \ldots\}$$

for the complex Fourier series.

We can think of these lists of functions $\{\phi_1(x), \phi_2(x), \phi_3(x), \ldots\}$ as infinite bases in a vector space of functions. A given function $f(x)$ on the interval $[0, 1]$ can be expanded in an infinite sum

$$f(x) = \sum_i a_i \phi_i(x)$$

for each of these sets. This is analgous to expanding a given vector with respect to various bases in a vector space.

To understand this analogy, think back to your last linear algebra course. A collection of vectors in a vector space form a basis if they span the space (that is, every vector can be written as a linear combination of basis vectors) and are linearly independent (that is, there is exaclty one way to write this linear combination). Three non-zero vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ in three dimensional space form a basis if they don't all lie in the same plane. In this case an arbitrary vector $\mathbf{v}$ can be expanded in a unique way as a linear combination

$$\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3$$

However, for a general basis, its not that easy to find the coefficients, a_1, a_2 and a_3. Finding them requires solving a system of linear equations.

Things are much easier if the basis is an orthogonal basis. This means that the inner (dot) product $\langle \mathbf{v}_i, \mathbf{v}_j \rangle$ is zero if $i \neq j$. In this case we can find the coefficients a_i by taking inner products. For example, to find a_1, we take the inner product with $\mathbf{v}_1$. This gives

$$\begin{aligned} \langle \mathbf{v}_1, \mathbf{v} \rangle &= a_1\langle \mathbf{v}_1, \mathbf{v}_1 \rangle + a_2\langle \mathbf{v}_1, \mathbf{v}_2 \rangle + a_3\langle \mathbf{v}_1, \mathbf{v}_3 \rangle \\ &= a_1\langle \mathbf{v}_1, \mathbf{v}_1 \rangle + 0 + 0 \end{aligned}$$

Thus

$$a_1 = \langle \mathbf{v}_1, \mathbf{v} \rangle / \langle \mathbf{v}_1, \mathbf{v}_1 \rangle$$

The sets of functions above can be thought of as infinite orthogonal bases for a vector space of functions. (I'm being a bit vague on exactly what functions are allowed, certainly all functions that are piecewise continuous with continuous derivative are included.)

What is the inner product of two functions $f(x)$ and $g(x)$? It is given by the integral

$$\langle f, g \rangle = \int_0^1 \bar{f}(x) g(x) dx$$

(The complex conjugate $\bar{f}$ is only relevant if f is complex valued. If f is real valued then $\bar{f} = f$.) With this definition of inner product each of the sets of functions are orthorgonal bases. This can be verified directly. For example, for n and m positive

$$\begin{aligned}\int_0^1 \sin(n\pi x)\sin(m\pi x)dx &= \frac{-1}{4}\int_0^1 (e^{in\pi x} - e^{-in\pi x})(e^{im\pi x} - e^{-im\pi x})dx \\ &= \frac{-1}{4}\int_0^1 e^{i(n+m)\pi x} - e^{i(n-m)\pi x} - e^{i(-n+m)\pi x} + e^{i(-n-m)\pi x}dx\end{aligned}$$

For any integer l

$$\int_0^1 e^{il\pi x} = \begin{cases} 1 & \text{if } l = 0 \\ \frac{e^{il\pi x}}{il\pi}\Big|_{x=0}^1 = \frac{(-1)^l - 1}{il\pi} & \text{if } l \neq 0 \end{cases}$$

Using this, its not hard to see that

$$\int_0^1 \sin(n\pi x)\sin(m\pi x)dx = \begin{cases} \frac{1}{2} & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

Now we can find the co-efficients in the sine expansion

$$f(x) = \sum b_n \sin(n\pi x)$$

directly, namely

$$b_n = \langle \sin(n\pi x), f(x)\rangle / \langle \sin(n\pi x), \sin(n\pi x)\rangle = 2\int_0^1 \sin(n\pi x)f(x)dx$$

Problem 2.11: Verify that $\mathbf{v}_1 = [1, 0, 1]$, $\mathbf{v}_2 = [1, 0, -1]$ and $\mathbf{v}_3 = [0, 1, 0]$ form an orthgonal bases. Find the coefficients a_i in the expansion $\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3$ when $\mathbf{v} = [1, 2, 3]$.

Problem 2.12: Expand the function $f(x) = 1$ in a sine series on the interval $0 \le x \le 1$.

Problem 2.13: Expand the function $f(x) = \sin(\pi x)$ in a cosine series on the interval $0 \le x \le 1$.

Symmetric operators, eigenvalues and orthogonal bases

There is a much more profound analogy between the infinite bases listed above and linear algebra. Let A be an $n \times n$ matrix. Recall that a vector $\mathbf{v}$ is called an eigenvector for A with eigenvalue λ if

$$A\mathbf{v} = \lambda\mathbf{v}$$

Recall that A is called *symmetric* (or, more generally, in the case of complex matrices, *hermitian*) if

$$\langle \mathbf{x}, A\mathbf{y}\rangle = \langle A\mathbf{x}, \mathbf{y}\rangle$$

(an equivalent condition is that $A^T = A$ (symmetric) or $\bar{A}^T = A$ (hermitian)). There is the following theorem.

Theorem 2.3 *If A is a symmetric (hermitian) $n \times n$ matrix then all the eigenvalues of A are real, and there is an orthogonal basis of eigenvectors.*

The infinite dimensional analogue of the matrix A will be the operator $-d^2/dx^2$—together with a boundary condition. (Later on we will also consider more general operators.) When A is a matrix and $\mathbf{v}$ is a vector, then A acts on $\mathbf{v}$ by matrix multiplication, producing the new vector $A\mathbf{v}$. When A is $-d^2/dx^2$ and $\phi(x)$ a function, then A acts on ϕ by differentiation, producing the new function $A\phi = \phi''$. In both cases the action of A is linear.

Going back to the infinite orthonormal bases in the last section, we can now see that they all consist of eigenfunctions of $-d^2/dx^2$. For example

$$-d^2/dx^2 \cos(2\pi nx) = (2\pi n)^2 \cos(2\pi nx)$$

so $\cos(2\pi nx)$ is an eigenfunction with eigenvalue $(2\pi n)^2$. (By the way, the reason for the minus sign in $-d^2/dx^2$ is to make the eigenvalues positive.) Similarly

$$-d^2/dx^2 e^{2\pi inx} = (2\pi n)^2 e^{2\pi inx}$$

and so on, for all the functions appearing.

Something seems fishy, though. Recall that for a matrix, any two eigenvectors corresponding to different eigenvalues are orthogonal. While this is true if we pick two eigenfunctions from the same list, it is not true if we pick one eigenfunction from one list and one from another. For example $\sin(\pi x)$ (eigenvalue π^2) is not orthogonal to 1 (eigenvalue 0) since $\int_0^1 \sin(\pi x) dx \neq 0$.

To explain this, lets try to check whether the operator $-d^2/dx^2$ is symmetric. Using integration by parts we find

$$\begin{aligned}
\langle f, -\frac{d^2}{dx^2} g\rangle &= -\int_0^1 f(x) g''(x) dx \\
&= -f(x)g'(x)\Big|_0^1 + \int_0^1 f'(x)g'(x)dx \\
&= -f(x)g'(x)\Big|_0^1 + f'(x)g(x)\Big|_0^1 - \int_0^1 f''(x)g(x)dx \\
&= -f(1)g'(1) + f(0)g'(0) + f'(1)g(1) - f'(0)g(0) + \langle -\frac{d^2}{dx^2} f, g\rangle
\end{aligned}$$

So we see that there are boundary terms spoiling the symmetry of the operator. To get these boundary term to disappear, we can impose boundary conditions on the functions f and g. For example we can impose

- *Dirichlet* boundary conditions: functions vanish at the endpoints, i.e., $f(0) = f(1) = g(0) = g(1) = 0$
- *Neumann* boundary conditions: derivatives vanish at the endpoints, i.e., $f'(0) = f'(1) = g'(0) = g'(1) = 0$
- *periodic* boundary conditions: functions (and derivatives) are periodic, i.e., $f(0) = f(1)$, $f'(0) = f'(1)$, $g(0) = g(1)$ and $g'(0) = g'(1)$

The imposition of any one of these boundary conditions makes $-d^2/dx^2$ a symmetric operator.

So the correct analog of a hermitian matrix A is not *just* the operator $-d^2/dx^2$, it is the operator $-d^2/dx^2$ *together with* suitable boundary condtions, like Dirichlet, Neumann or periodic boundary conditions. (I'm saying exactly what constitutes a suitable set of boundary conditions here, and am sweeping some technical points under the rug.)

If we insist that all the eigenfunctions obey the same boundary conditions (Dirichlet, Neumann or periodic), then it is true that all the eigenvalues are real, and eigenfunctions corresponding to different eigenvalues are orthogonal. The proofs are exactly the same as for matrices.

Now let us try to determine the eigenfunctions and eigenvalues of $-d^2/dx^2$ with *Dirichlet* boundary conditions on the interval $[0, 1]$.

Theorem 2.4 *The eigenfunctions of $-d^2/dx^2$ with Dirichlet boundary conditions on the interval $[0, 1]$ are the functions $\sin(\pi x), \sin(2\pi x), \sin(3\pi x), \ldots$.*

Proof: We want to determine all functions $\phi(x)$ satisfying

$$-\phi''(x) = \lambda\phi(x) \tag{2.10}$$

for some λ and obeying Dirichlet boundary conditions. We know that λ must be real. Suppose that λ is negative. Then $\lambda = -\mu^2$ and the general solution to (2.10) is

$$\phi(x) = ae^{\mu x} + be^{-\mu x}$$

for arbitrary constants a and b. If we insist that the $\phi(0) = \phi(1) = 0$, then

$$\begin{aligned} 0 &= a + b \\ 0 &= ae^{\mu} + be^{-\mu} \end{aligned}$$

This system of linear equations has no solutions other than the trivial solution $a = b = 0$. Thus there are no eigenfunctions with negative λ (the zero function doesn't qualilfy, just as the zero vector doesn't count as an eigenvector for a matrtrix.)

Next we try $\lambda = 0$. In this case

$$\phi(x) = ax + b$$

for arbitrary constants. If we insist that $\phi(0) = 0$ then $b = 0$. Then $\phi(1) = a$, so $\phi(1) = 0$ forces $a = 0$ too. Thus $\lambda = 0$ is not an eigenvalue.

Finally we try positive λ. Then $\lambda = \mu^2$ and

$$\phi(x) = a\sin(\mu x) + b\cos(\mu x)$$

for arbitrary constants a and b. Now $\phi(0) = b$ so if we impose $\phi(0) = 0$ then $b = 0$ and $\phi(x) = a\sin(\mu x)$. If we further insist that $\phi(1) = 0$, then $a\phi(\mu) = 0$. This can happen in two ways. Either $a = 0$, in which case $\phi = 0$. This we don't want. But $a\phi(\mu) = a\sin(\mu)$. This will be zero if $\mu = n\pi$ for an integer n. We can rule out $n = 0$, since this implies $\phi = 0$. Two eigenfunctions are counted as the same if they are multiples of each other. Since $\sin(-n\pi) = -\sin(n\pi)$, n and $-n$ give the same eigenfunctions, and so we may assume $n > 0$. Also we can set $a = 1$. Thus the eigenfunctions are

$$\sin(\pi x), \sin(2\pi x), \sin(3\pi x), \ldots$$

as claimed. ☐

Problem 2.14: Show that the eigenvalues of a symmetric operator (or matrix) are real, and that eigenfunction (or eigenvectors) corresponding to different eigenvalues are orthogonal. (You can find these proofs in any linear algebra text.)

Problem 2.15: Show that the functions $1, \cos(\pi x), \cos(2\pi x), \cos(3\pi x), \ldots$ are the eigenfunctions of $-d^2/dx^2$ with Neumann boundary conditions.

The infinite bases

$$\{1, \cos(2\pi x), \cos(4\pi x), \cos(6\pi x), \ldots \sin(2\pi x), \sin(4\pi x), \sin(6\pi x), \ldots\}$$

and

$$\{\ldots, e^{-4i\pi x}, e^{-2i\pi x}, e^{0i\pi x}, e^{2i\pi x}, e^{4i\pi x}, \ldots\}$$

are both orthogonal bases for $-d^2/dx^2$ with periodic boundary conditions. In this case each eigenvalue has multiplicity 2.

The complex basis functions $\{\ldots, e^{-4i\pi x}, e^{-2i\pi x}, e^{0i\pi x}, e^{2i\pi x}, e^{4i\pi x}, \ldots\}$ are special, because not only are they eigenfunctions for the second derivative $-d^2/dx^2$, but also for the first derivative d/dx.

Smoothness and decrease of Fourier coefficients

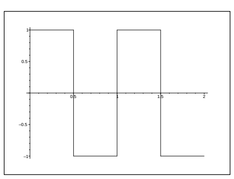
Consider a function $f(x)$, periodic with period 1 with Fourier series

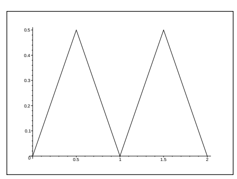
$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x}$$

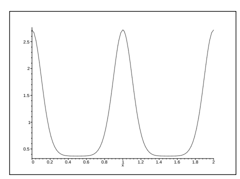
All the information about the function f must somehow be encoded in the Fourier coefficients, since knowing the c_i's is equivalent to knowing the function f. One property of the function which is easy to read off from the Fourier coefficients is the degree of smoothness. Rough or discontinuous functions will have Fourier coefficients c_n that become small very slowly as $|n|$ becomes large. On the other hand, smoothly varying functions will have c_n which tend to zero very quickly as $|n|$ becomes large.

Lets see how this works in some examples. Lets consider the three functions f, g and h given by

$$f(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1/2 \\ -1 & \text{if } 1/2 < x \le 1 \end{cases} \quad g(x) = \begin{cases} x & \text{if } 0 \le x \le 1/2 \\ 1-x & \text{if } 1/2 < x \le 1 \end{cases} \quad h(x) = e^{\cos(2\pi x)} \cos(\sin(2\pi x))$$





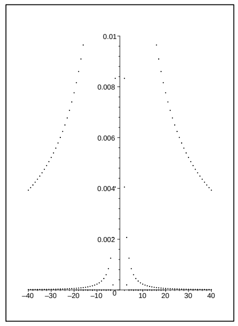


Notice that f is not continuous, g is continuous but not differentiable, and h is very smooth, and can be differentiated any number of times. Now lets consider the Fourier coefficients of these functions. They are give by

$$f \qquad\qquad\qquad g \qquad\qquad\qquad h$$

$$c_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ 2/(i\pi n) & \text{if } n \text{ is odd} \end{cases} \quad c_n = \begin{cases} 1/4 & \text{if } n = 0 \\ 0 & \text{if } n \text{ is even and } n \neq 0 \\ -1/(\pi^2 n^2) & \text{if } n \text{ is odd} \end{cases} \quad c_n = \begin{cases} 1 & \text{if } n = 0 \\ 1/(|n|!) & \text{if } n \neq 0 \end{cases}$$

Notice that the Fourier coefficients for f decrease like $1/n$, the coefficients for g decrease more quickly, like $1/n^2$ and the coefficients for h decrease extremely rapidly, like $1/|n|!$. Here is a plot of the size $|c_n|$ as a function of n for these three cases. (I have not plotted the zero coefficients).



Why is it that smooth functions have smaller coefficients for large $|n|$? One explanation is that the basis functions $e^{2\pi nx}$ are oscillating more quickly as n gets large. If f is itself a wildly changing function, then large doses of the quickly oscillating basis functions are needed to reconstruct f as a Fourier series.

Another way to understand this is to consider what happens when you differentiate a Fourier series. Suppose that

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi inx}$$

Then when we differentiate, then (assuming we can exchange the order of summation and differentiation), we get

$$\begin{aligned}\frac{d}{dx}f(x) &= \sum_{n=-\infty}^{\infty} c_n \frac{d}{dx} e^{2\pi inx}\\ &= \sum_{n=-\infty}^{\infty} 2\pi inc_n e^{2\pi inx}\end{aligned}$$

Here we used the fact the function $e^{2\pi inx}$ is an eigenfunction for differentiation. So the Fourier coefficients of f' are $2\pi inc_n$, i.e., just the original coefficients multiplied by $2\pi in$. Similarly

$$\frac{d^k}{dx^k}f(x) = \sum_{n=-\infty}^{\infty} (2\pi in)^k c_n e^{2\pi inx}$$

These formulas are valid provided the series on the right converge. Note however that the factor $(2\pi in)^k$ is growing large when n increases. So the series for the derivative will only converge if the coefficients c_n are decreasing quickly enough to compensate for the growth of $(2\pi in)^k$.

We will examine a practical application of this idea, but first we must discuss Fourier series in higher dimensions.

Fourier series in 2 and 3 dimensions

Suppose that $f(x, y)$ is a function of two variables defined on the square $0 \le x \le 1$, $0 \le y \le 1$. Then f can be expanded in a double Fourier series. Think of first fixing y. Then for this fixed y, $f(x, y)$ is a function of x that can be expanded in a Fourier series. However, the coefficients will depend on y. Thus we obtain

$$f(x, y) = \sum_{n=-\infty}^{\infty} c_n(y) e^{2\pi inx}$$

Now each coefficient $c_n(y)$ is a function of y which can be expanded

$$c_n(y) = \sum_{m=-\infty}^{\infty} c_{n,m} e^{2\pi i m y}$$

If we combine these formulas we get a double expansion for f

$$f(x,y) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c_{n,m} e^{2\pi i n x} e^{2\pi i m y} = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c_{n,m} e^{2\pi i (nx+my)}$$

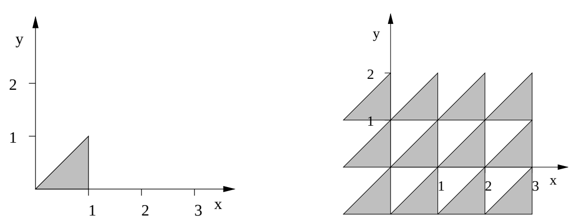
What is the formula for the $c'_{n,m}s$? Well

$$\begin{aligned} c_{n,m} &= \int_0^1 e^{-2\pi i m y} c_n(y) dy \\ &= \int_0^1 e^{-2\pi i m y} \int_0^1 e^{-2\pi i n x} f(x,y) dx dy \\ &= \int_0^1 \int_0^1 e^{-2\pi i (nx+my)} f(x,y) dx dy \end{aligned}$$

Lets find the coefficients for $f(x,y)$ defined on the unit square as

$$f(x,y) = \begin{cases} 0 & \text{if } y > x \\ 1 & \text{if } x \geq y \end{cases}$$

and then extended periodically with period 1 in both directions. Here is a picture of f and its periodic extension. The shaded area is where f is equal to 1 and the white area where f is 0.



We have

$$c_{n,m} = \int_0^1 \int_y^1 e^{-2\pi i (nx+my)} dx dy$$

If $n = m = 0$ then

$$c_{0,0} = \int_0^1 \int_y^1 dx dy = 1/2$$

If $n = 0$ and $m \neq 0$ then (changing the order of integration)

$$\begin{aligned} c_{0,m} &= \int_0^1 \int_0^x e^{-2\pi i m y} dy dx \\ &= \int_0^1 \frac{e^{-2\pi i m y}}{-2\pi i m} \Big|_{y=0}^{x} dx \\ &= \frac{1}{-2\pi i m} \int_0^1 (e^{-2\pi i m x} - 1) dx \\ &= \frac{1}{2\pi i m} \end{aligned}$$

If $n \neq 0$ and $m = 0$ then

$$\begin{aligned} c_{n,0} &= \int_0^1 \int_y^1 e^{-2\pi nx} dxdy \\ &= \int_0^1 \frac{e^{-2\pi inx}}{-2\pi in}\Big|_{x=y}^1 dy \\ &= \frac{1}{-2\pi in} \int_0^1 (1 - e^{-2\pi iny}) dy \\ &= \frac{1}{-2\pi in} \end{aligned}$$

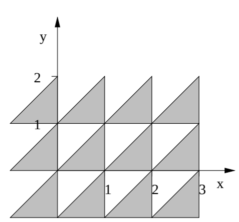
If $n + m = 0$ (i.e., $m = -n$) but $n \neq 0$ then

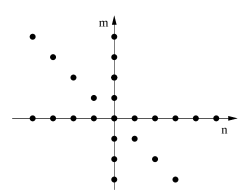
$$\begin{aligned} c_{n,-n} &= \int_0^1 \int_y^1 e^{2\pi iny} e^{-2\pi inx} dxdy \\ &= \int_0^1 e^{2\pi ny} \frac{e^{-2\pi inx}}{-2\pi in}\Big|_{x=y}^1 \\ &= \frac{1}{-2\pi in} \int_0^1 e^{2\pi ny}(1 - e^{-2\pi iny}) dy \\ &= \frac{1}{-2\pi in} \int_0^1 (e^{2\pi ny} - 1) dy \\ &= \frac{1}{2\pi in} \end{aligned}$$

Otherwise (i.e., $n \neq 0$, $m \neq 0$, $n + m \neq 0$)

$$\begin{aligned} c_{n,m} &= \int_0^1 \int_y^1 e^{-2\pi imy} e^{-2\pi inx} dxdy \\ &= \int_0^1 e^{-2\pi imy} \frac{e^{-2\pi inx}}{-2\pi in}\Big|_{x=y}^1 \\ &= \frac{1}{-2\pi in} \int_0^1 e^{-2\pi imy} - e^{-2\pi i(m+n)y} \\ &= 0 \end{aligned}$$

It is interesting to relate the directions in n, m space where the Fourier coefficients are large to the directions where f has discontinuities. In this example, most of the $c_{m,n}$'s are zero. But in the places that they are non-zero, they are quite large. Here is a picture of f together with a picture of where the $c_{n,m}$'s are non-zero.





Notice how each cliff-like discontinuity in f produces a line of large $c_{n,m}$'s at right angles to the cliff. And notice also that you have to consider discontinuities in the periodic extension of f (i.e., you have to take into account the horizontal and vertical cliffs, and not just the diagonal one.)

We can consider double Fourier sine series and cosine series as well (and even mixtures, doing one expansion in one direction and one expansion in the other). For the double sine series, we obtain

$$f(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_{n,m}\sin(\pi nx)\sin(\pi my)$$

with

$$b_{n,m} = 4\int_0^1\int_0^1 \sin(\pi nx)\sin(\pi my)f(x,y)dxdy$$

and the analogous formula holds for the cosine series.

For the double sine series, the functions $\phi_{n,m}(x,y) = \sin(\pi nx)\sin(\pi my)$ can be considered to be eigenfunctions of (minus) the Laplace operator with Dirichlet (zero) boundary conditions on the boundary of the square. This is because when x is 0 or 1 or y is 0 or 1 (i.e., on the boundary of the square), $\phi_{n,m}(x,y) = 0$ and

$$\begin{aligned}-\Delta\phi_{n,m}(x,y) &= -(\partial^2/\partial x^2 + \partial^2/\partial y^2)\sin(\pi nx)\sin(\pi my)\\ &= \pi^2(n^2+m^2)\sin(\pi nx)\sin(\pi my)\end{aligned}$$

Problem 2.16: Show that the function $\phi(x,y) = e^{2\pi i(nx+my)}$ is an eigenfunction of (minus) the Laplace operator in two dimensions (i.e., $-\Delta = -\partial^2/\partial x^2 - \partial^2/\partial y^2$) satisfying periodic boundary conditions in both directions (i.e., $\phi(x+1,y) = \phi(x,y)$ and $\phi(x,y+1) = \phi(x,y)$. What is the eigenvalue?

Problem 2.17: Expand the function defined on the unit square $0 \le x \le 1$, $0 \le y \le 1$ by

$$f(x,y) = \begin{cases} 0 & \text{if } y > x\\ 1 & \text{if } x \ge y\end{cases}$$

in a double sine series

Problem 2.18: Expand the same function in a double cosine series

Problem 2.19: Expand the function defined on the unit square $0 \le x \le 1$, $0 \le y \le 1$ by

$$f(x,y) = \begin{cases} 1 & \text{if } x < 1/2 \text{ and } y < 1/2\\ 0 & \text{otherwise}\end{cases}$$

in a double complex Fourier series.

The discrete Fourier transform

Suppose that we don't know the function f everywhere on the interval, but just at N equally spaced discrete points $0, 1/N, 2/N, \ldots, (N-1)/N$. Define $f_j = f(j/N)$ Then we can write down an approximation for c_n by using the Riemann sum in place of the integral. Thus we define the c_k's for the discrete Fourier transform to be

$$c_k = \frac{1}{N}\sum_{j=0}^{N-1} e^{-i2\pi kj/N} f_j$$

The first thing to notice about this is that although the formula makes sense for all k, the c_k's start repeating themselves after a while. In fact $c_{k+N} = c_k$ for all k. This follows from the fact that $e^{-i2\pi j} = 1$ which implies that $e^{-i2\pi(k+N)j/N} = e^{-i2\pi kj/N}e^{-i2\pi j} = e^{-i2\pi kj/N}$, so the formulas for c_k and c_{k+N} are the same. So we might as well just compute $c_0, \ldots, c_{N-1}$.

Next, notice that the transformation that sends the vector $[f_0, \ldots, f_{N-1}]$ to the vector $[c_0, \ldots, c_{N-1}]$ is a linear transformation, given by multiplication by the matrix $F = [F_{k,j}]$ with $F_{k,j} = \frac{1}{N}e^{-i2\pi kj/N}$. If we define $w = e^{-i2\pi/N}$ then the matrix has the form

$$F = \frac{1}{N}\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w & w^2 & \cdots & w^{N-1} \\ 1 & w^2 & w^4 & \cdots & w^{2(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & w^{N-1} & w^{2(N-1)} & \cdots & w^{(N-1)(N-1)} \end{bmatrix}$$

To compute the inverse of F we use the following fact about roots of unity. A complex number z is a Nth root of unity if $z^N = 1$ of $z^N - 1 = 0$. There are N such numbers, given by $1, e^{-i2\pi/N}, e^{-i2\pi 2/N}, \ldots, e^{-i2\pi(N-1)/N}$, or $1, w, w^2, \ldots, w^{N-1}$. The following factorization

$$z^N - 1 = (z-1)(1 + z + z^2 + \cdots + z^{N-1})$$

(which you can check by just multiplying out the right side) implies that for any Nth root of unity z that is different from 1, we have

$$(1 + z + z^2 + \cdots + z^{N-1}) = 0. \tag{2.11}$$

To see this simply plug z into the factorization. Then the left side is $z^N - 1 = 0$, but $(z-1)$ isn't zero, so we may divide by $(z-1)$.

Using (2.11) we can now see that the inverse to F is given by

$$F^{-1} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \overline{w} & \overline{w}^2 & \cdots & \overline{w}^{N-1} \\ 1 & \overline{w}^2 & \overline{w}^4 & \cdots & \overline{w}^{2(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \overline{w}^{N-1} & \overline{w}^{2(N-1)} & \cdots & \overline{w}^{(N-1)(N-1)} \end{bmatrix}$$

where $\overline{w}$ is the complex conjugate of w given by $\overline{w} = e^{i2\pi/N} = 1/w$. So we see that the matrix for the inverse transform is the same, except that the factor of $1/N$ is missing, and i is replaced with $-i$.

The fast Fourier transform (FFT)

If you count how many multiplications need to be done when multiplying a vector of length N with and N by N matrix, the answer is N^2. So it would be reasonable to assume that it takes N^2 multiplications to compute the discrete Fourier transform. In fact, if N is a power of 2, it is possible to do the computation in using only $N\log(N)$ multiplications. This is a huge improvement, and any practical applications of the Fourier transform on computers will make use of the FFT. I won't have time to discuss the fast Fourier transform in this course, but I have prepared some notes for Math 307 that are make available on my web page.

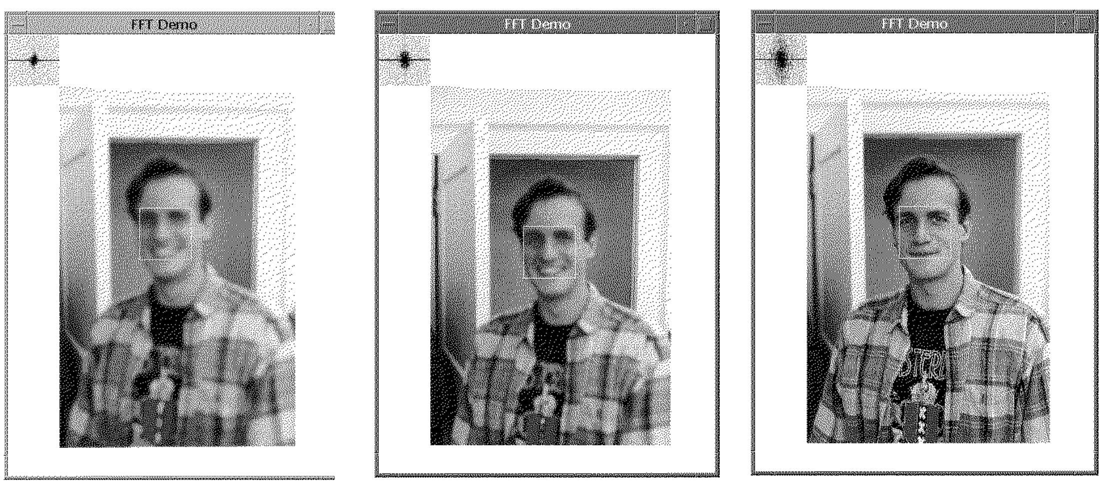
An application: which picture is in focus?

We can consider a a black and white image to be a function $f(x, y)$ of two variables. The value $f(x, y)$ gives the intensity at the point (x, y). For a colour picture there are three functions, one for each of red, green and blue. Of course, for a digital image, these functions are only defined for a discrete set of points, namely the pixels.

We can define the discrete two dimensional Fourier transform analogously to the one-dimensional case considered above. The Fourier coefficients $c_{n,m}$ then depend on two numbers, n and m.

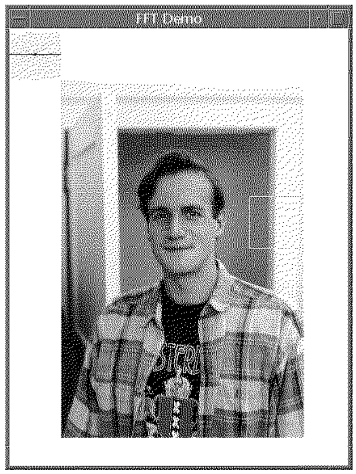
There is a relation between the size of $|c_{n,m}|$ for large n and m and the sharpness of the image. If the image is sharp, then the function f will not be very smooth, but have rapid changes in intensities. Thus the coefficients for large n and m. will be large. This fact can be used to focus a camera. The only slightly tricky point for the discrete Fourier transform is to decide which n's and m's count as large. One might think that $N - 1$ is a large index, but $c_{N-1} = c_{-1}$, (we're back in the one dimensional case here) and -1 is close to zero, i.e., pretty small. So the "largest" n's and m's are the ones near $N/2$. To reflect this fact, its better to plot the $|c_{n,m}|$'s for n and m ranging from $-N/2$ to $N/2 - 1$.

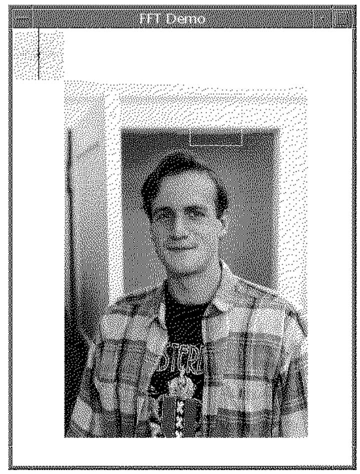
Here are three photos. The discrete Fourier transform of the 64×64 outlined window (the outline might be a bit hard to see) is shown in the top left corner. For the Fourier transforms, the darkness indicates the size of $|c_{n,m}|$. Here we can see that the picture most in focus (on the right) has the most spread out region of Fourier coefficients.



But what are those dark horizontal and vertical lines in the Fourier transform? This can be explained by the fact that when we take the Fourier transform of a 64×64 subregion, we are in fact repeating the region periodically over the whole plane.

This will introduce artificial horizontal and vertical discontinuities at the edges. The result will be large Fourier coefficients in the directions normal to these discontinuities. To test this, lets place the window on a place where the top of the picture is the same as the bottom. Then, when we tile the plane, the discontinuities will come only from the sides, and should result in large Fourier coefficients only in the normal (horizontal) direction. Similarly, if the window is moved to where the picture is the same on the left and the right side, the Fourier coefficients are large only in the vertical direction.





From the point of view of the original picture, these horizontal and vertical discontinuities are artificial. We can improve the situation by taking the (double) cosine transform. This is because, just as in the one dimensional case, the cosine transform can be thought of as the Fourier transform of the picture after reflecting it to make an even function. The even function will not have any discontinuities that are not present in the original picture.

Image compression

The (discrete) Fourier transform can also be used for image compression. Recall that the Fourier coefficients $c_{-N/2,-N/2}, \ldots, c_{N/2-1,N/2-1}$ contain all the information about the picture. So if we store all the coefficients, we will be able to reconstruct the picture perfectly using the Fourier formula

$$f_{j,k} = \sum_{n=-N/2}^{N/2-1} \sum_{m=-N/2}^{N/2-1} c_{n,m} e^{2\pi i(jn+km)/N}$$

In practice, even for an image in focus, the $c_{n,m}$'s for large n and m will be quite small. So instead of storing all the $c_{n,m}$'s, we store just 75% or 50% of them. Then, when reconstructing the picture, we simply set the $c_{n,m}$'s whose value we don't know equal to zero.

This idea is behind some of the image compression schemes in use. There are a few more wrinkles, though. First of all, the cosine transform is used, for the reason explained above. Secondly, instead of taking the transform of the whole picture, the picture is first tiled into small windows, and the transform of each window is computed.

The heat equation

We will now consider the heat equation. The approach we will take to solving follows closely the ideas used in solving the vector equation

$$\mathbf{v}'(t) = -A\mathbf{v}(t)$$

with initial condition $\mathbf{v}(0) = \mathbf{v}_0$, where A is an $n \times n$ matrix. The idea is to expand $\mathbf{v}(t)$ in a basis of eigenvectors of A. The coefficients in the expansion will depend on t. Plugging the expansion into the equation yields an ordinary differential equation for each expansion coefficient. This equation has a solution that depends on an arbitrary constant. These arbitrary constants are then adjusted so that the initial condition is fulfilled.

In the case of heat equation the operator A is replaced by $-\Delta$ together with a boundary condition. The expansions in eigenfunctions will be a Fourier expansion. The new wrinkle is the presence of boundary conditions.

One space dimension with zero boundary conditions

We begin with the simplest situation: one space dimension and zero boundary conditions. The experimental setup is a long thin rod, insulated except at the ends. The ends of the rod are both kept at a constant temperature of $0°$. At time $t = 0$ we are given the initial temperature distribution of the rod. The goal is to find the temperature at all later times.

Let x denote the distance along the rod, which we assume varies between 0 and a. Let $u(x, t)$ for $0 \le x \le a$ and $t \ge 0$ denote the temperature of the rod at position x and time t. Then $u(x, t)$ will satisfy the heat equation in one dimension

$$\frac{\partial u(x,t)}{\partial t} = \alpha^2 \frac{\partial^2 u(x,t)}{\partial x^2},$$

the boundary condition

$$u(0, t) = u(a, t) = 0 \quad \text{for all } t > 0$$

and the initial condition

$$u(x, 0) = u_0(x)$$

where $u_0(x)$ is the given initial temperature distribution. The number α^2 is determined by the material of which the rod is made.

For every fixed t, we now expand this unknown function $u(x, t)$ in a series of eigenfunctions of $-\frac{\partial^2}{\partial x^2}$. Since we want $u(x, t)$ to satisfy zero boundary conditions, we choose eigenfunctions that also satisfy zero (Dirichlet) boundary conditions. These are the sine functions $\{\sin(n\pi x/a)\}$ for $n = 1, 2, 3, \ldots$. Thus the expansion will be a sine series. The (unknown) coefficients will be functions of t. Thus

$$u(x, t) = \sum_{n=1}^{\infty} \beta_n(t) \sin(n\pi x/a)$$

To determine u we must determine the coefficients $\beta_n(t)$.

We now plug the expansion into the heat equation. Since

$$\frac{\partial u(x,t)}{\partial t} = \sum_{n=1}^{\infty} \beta_n'(t) \sin(n\pi x/a)$$

and (here we use the crucial fact that the sine functions are eigenfunctions)

$$\frac{\partial^2 u(x,t)}{\partial x^2} u(x,t) = \sum_{n=1}^{\infty} -(n\pi/a)^2 \beta_n(t) \sin(n\pi x/a)$$

the equation will be satisfied if

$$\beta_n'(t) = -\alpha^2 (n\pi/a)^2 \beta_n(t)$$

This ODE has solution

$$\beta_n(t) = b_n e^{-\alpha^2 (n\pi/a)^2 t},$$

where b_n is an arbitrary constant. The function

$$\sum_{n=1}^{\infty} b_n e^{-\alpha^2 (n\pi/a)^2 t} \sin(n\pi x/a)$$

will satisfy the heat equation and the boundary conditions for any choice of the b_n's. However, we still need to satisfy the initial condition. The initial condition will hold if

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin(n\pi x/a) = u_0(x)$$

In other words, the b_n are the coefficients of $u_0(x)$ in a Fourier sine series. We know how to find these (we did it for $a = 1$ before). The formula is

$$b_n = \frac{2}{a} \int_0^a \sin(n\pi x/a) u_0(x) dx$$

We have now determined $u(x,t)$ completely.

Example

Suppose we join a rod of length 1 and constant temperature 100° with a rod of length 1 and constant temperature 0°. Thereafter, the two ends of the joined rod are kept at 0°. Both rods are made of the same metal with $\alpha^2 = 1$. Lets find the temperature function $u(x,t)$.

Since the ends at $x = 0$ and $x = 2$ are kept at 0°, u will have an expansion

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\alpha^2 (n\pi/2)^2 t} \sin(n\pi x/2).$$

The initial condition is

$$u_0(x) = \begin{cases} 100 & \text{if } 0 \le x \le 1 \\ 0 & \text{if } 1 \le x \le 2 \end{cases}$$

Thus

$$\begin{aligned} b_n &= \frac{2}{2} \int_0^2 \sin(n\pi x/2) u_0(x) dx \\ &= 100 \int_0^1 \sin(n\pi x/2) \\ &= \frac{200}{n\pi} (\cos(n\pi x/2) - 1) \end{aligned}$$

Smoothness of solutions to the heat equation

Recall that the smoothness of a function is encoded in the rate of decrease of the size of the Fourier coefficients as n gets large. For solutions to the heat equation, the Fourier coefficients have the form $b_n e^{-\alpha^2(n\pi/a)^2 t}$. Now the coefficients b_n will typically have some decay, corresponding to the smoothness of the initial condition. But as soon as $t > 0$ there is an extra factor $e^{-\alpha^2(n\pi/a)^2 t}$ that decays extremely rapidly as n gets large. This means that any roughness in the initial temperature distribution will be immediately smoothed out once heat begins to flow.

(Later, when we study the wave equation, we will see that the situation is quite different there. In that case the Fourier coefficients of the moving wave decay at about the same rate as the coeficients of the initial condition. So for the wave equation there is no smoothing effect. In fact, a "corner" in the solution, say a wave crest, will travel with time. Studying this motion has been one of the most intensely studied questions in PDE in this century. It goes under the name of "propagation of singularities" or, in the case of light waves, "the theory of geometrical optics.")

Steady state solutions and non-homogeneous boundary conditions

Mathematically, zero boundary conditions are natural because they are homogeneous. This means that if you form a linear combination of functions with the boundary conditions, the resulting function still satisfies the boundary conditions. However, from the physical point of view it is absurd that there should be anything special about holding the ends of the rod at $0°$, since the zero point on the temperature scale is completely arbitrary.

We will now see how to solve the heat flow problem in a thin rod when the ends are held a any fixed temperatures. We begin with the notion of a steady state solution $\varphi(x)$. This is a solution of the heat equation that doesn't depend on time. Thus

$$0 = \frac{\partial \varphi}{\partial t} = \alpha^2 \frac{\partial^2 \varphi}{\partial x^2},$$

or,

$$\varphi''(x) = 0$$

This equation is easy to solve (because we are in one space dimension) and we obtain

$$\varphi(x) = ax + b$$

for constants a and b.

Now notice that if $u(x,t)$ solves the heat equation, then so does $u(x,t) - \varphi(x)$. This is because the heat equation is a linear equation. However $u(x,t) - \varphi(x)$ satisfies different boundary conditions. Thus we may use φ to adjust the boundary conditions.

Suppose we wish to solve

$$\frac{\partial u(x,t)}{\partial t} = \alpha^2 \frac{\partial^2 u(x,t)}{\partial x^2},$$

but now with the non-homogeneous boundary condition

$$u(0,t) = A, \quad u(a,t) = B \quad \text{for all } t > 0$$

and, as before, the initial condition

$$u(x,0) = u_0(x)$$

First we find an equilibrium solution satisfying the same non-homogeneous boundary conditions.

$$\varphi(x) = A + \frac{B - A}{a} x$$

Let $v(x,t)$ be the difference

$$v(x,t) = u(x,t) - \varphi(x).$$

Then $v(x,t)$ still satisfies the heat equation. But the boundary conditions satisfied by $v(x,t)$ are

$$v(0,t) = u(0,t) - \varphi(0) = A - A = 0$$

and

$$v(a,t) = u(a,t) - \varphi(a) = B - B = 0$$

In other words, v satisfies Dirichlet boundary conditions. The initial conditions satisfied by v are

$$v(x,0) = u(x,0) - \varphi(0) = u_0(x) - \varphi(x)$$

We can now find $v(x,t)$ using the sine series and then set $u(x,t) = v(x,t) + \varphi(x)$.

Problem 3.1: Suppose a rod of length 1 with $\alpha^2 = 1$ is held with one end in a heat bath at 0° and the other end in a heat bath at 10° for a long time, until a steady state is reached. Then, after $t = 0$, both ends are kept at 0°. Find the temperature function.

Problem 3.2: You are a detective working on the case of the missing gold bars. These thin bars are 100 cm long and were carefully wrapped, except for the ends, in an insulating blanket. When you find a bar in the snow (at 0°) you quickly take out your thermometer and measure the temperature at the centre of the bar to be 0.3°. Assuming the bar fell out of the getaway car (at 20°), how long has it been lying there? (In this story, α^2 for gold is $1.5\mathrm{cm}^2/\mathrm{sec}$)

Problem 3.3: Suppose a rod of length 1 with $\alpha^2 = 1$ is held with one end in a heat bath at 0° and the other end in a heat bath at 10° for a long time, until a steady state is reached. Then, after $t = 0$, the heat baths are switched. Find the temperature function.

Insulated ends

If $u(x,t)$ is the temperature function for an insulated rod, then $\partial u(x,t)/\partial x$ represents the heat flux through a cross section at position x at time t. If we insulate the ends of the rod, then the heat flux is zero at the ends. Thus, for an insulated rod of length a, we have $\partial u(0,t)/\partial x = \partial u(a,t)/\partial x = 0$. These are the boundary condition for an insulated rod.

So to find the temperature function $u(x,t)$ for an insulated rod, insulated also at each end, with initial temperature distribution $u_0(x)$, we must solve

$$\frac{\partial u(x,t)}{\partial t} = \alpha^2 \frac{\partial^2 u(x,t)}{\partial x^2},$$

with the boundary condition

$$\frac{\partial u(0,t)}{\partial x} = \frac{\partial u(a,t)}{\partial x} = 0 \quad \text{for all } t > 0$$

and the initial condition

$$u(x,0) = u_0(x)$$

Clearly the right thing to do in this case is to expand $u(x,t)$ in a cosine series, since these are the eigenfunctions of $-\partial^2/\partial x^2$ satisfying the boundary conditions. Thus we write

$$u(x,t) = \alpha_0(t)/2 + \sum_{n=1}^{\infty} \alpha_n(t)\cos(n\pi x/a)$$

Then $u(x,t)$ will automatically satisfy the correct boundary conditions. When we plug this into the heat equation, we obtain, as before, an ODE for each $\alpha_n(t)$:

$$\alpha_0'(t)/2 = 0$$

and

$$\alpha_n'(t) = -\alpha^2(n\pi/a)^2\alpha_n(t)$$

Thus

$$\alpha_0(t) = a_0$$

and

$$\alpha_n(t) = a_n e^{-\alpha^2(n\pi/a)^2 t},$$

where the a_n's are arbitrary constants. To determine the a_n's we use the initial condition

$$u(x,0) = a_0/2 + \sum_{n=1}^{\infty} a_n\cos(n\pi x/a) = u_0$$

and find the the a_n's are the coefficients of u_0 in a cosine expansion. Thus

$$a_n = \frac{2}{a}\int_0^a \cos(n\pi x/a)u_0(x)dx$$

and the solution is completely determined.

Problem 3.4: Show that after a long time, the temperature in the rod is uniform, equal to the average of the initial temperature distribution.

Problem 3.5: Find the temperature function of an insulated rod of length 1, insulated at both ends, if the initial temperature distribution is $u_0(x) = x$.

Non-homogeneous boundary conditions involving the derivative

Suppose we replace the homogeneous boundary condition in the last section with

$$\frac{\partial u(0,t)}{\partial x} = A, \quad \frac{\partial u(a,t)}{\partial x} = B \quad \text{for all } t > 0$$

Physically this means that heat is being pumped in or out (depending on the sign of A and B) at the endpoints.

We can try to mimic what we did before, and subtract off a steady state solution satisfying these boundary conditions. But steady state solutions are of the form $\varphi(x) = mx + b$ so $\varphi'(x) = m$. Unless $A = B$ it is not possible that $\varphi'(0) = A$ and $\varphi'(a) = B$.

Physically, this makes sense. If we are pumping in a certain amount of heat from the right, and are pumping out a different amount, then there will be no steady state solution. Heat will either build up or be depleted indefinitely.

So instead of subtracting off a steady state solution, we will subtract off a particular solution that depends on both x and t. Let

$$\varphi(x,t) = bx^2 + cx + dt.$$

Then

$$\frac{\partial \varphi(x,t)}{\partial t} = d$$

and

$$\alpha^2 \frac{\partial^2 \varphi(x,t)}{\partial x^2} = 2\alpha^2 b$$

So if we set $d = 2\alpha^2 b$ then $\varphi(x,t)$ solves the heat equation. Now $\partial\varphi(x,t)/\partial x = 2bx + c$ and we may adjust b and c so that φ satisfies the desired non-homogeneous boundary conditions. Letting $c = A$ and $b = (B-A)/2a$ we find $= \partial\varphi(0,t)/\partial x = A$ and $\partial\varphi(a,t)/\partial x = B$. The function $\varphi(x,t)$ is completely determined by these choices, namely

$$\varphi(x,t) = (B-A)x^2/2a + Ax + \alpha^2(B-A)t/a.$$

Now we form the new function $v(x,t) = u(x,t) - \varphi(x,t)$. The function $v(x,t)$ will solve the heat equation, being the difference of two solutions. The boundary conditions satisfied by $v(x,t)$ is

$$\frac{\partial v(0,t)}{\partial x} = \frac{\partial u(0,t)}{\partial x} - \frac{\partial \varphi(0,t)}{\partial x} = A - A = 0, \quad \frac{\partial v(a,t)}{\partial x} = \frac{\partial u(a,t)}{\partial x} - \frac{\partial \varphi(a,t)}{\partial x} = B - B = 0 \quad \text{for all } t > 0$$

and the initial condition satisfied by $v(x,t)$ is

$$v(x,0) = u(x,0) - \varphi(x,0) = u_0(x) - (B-A)x^2/2a + Ax$$

Therefore we can solve for $v(x,t)$ as a cosine series, and the solution will be given by $u(x,t) = v(x,t) + \varphi(x,t)$.

Problem 3.6: Solve the heat equation

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2},$$

for a rod of length 1, with the non-homogeneous boundary condition

$$\frac{\partial u(0,t)}{\partial x} = 2, \quad \frac{\partial u(1,t)}{\partial x} = 1 \quad \text{for all } t > 0$$

and initial condition

$$u(x,0) = 0$$

Non-homogeneous term $f(x)$ in the equation

We have seen that in order to deal with non-homogeneous boundary conditions, we must subtract off a particular solution of the heat equation satisfying those boundary conditions. The same principle can be applied when there are non-homogeneous terms in the equation itself.To start, we will examine a case where the non-homogeneous term doesn't depend on t. Consider the equation

$$\frac{\partial u(x,t)}{\partial t} = \alpha^2 \frac{\partial^2 u(x,t)}{\partial x^2} + f(x),$$

with the non-homogeneous boundary condition

$$u(0,t) = A, \quad u(a,t) = B \quad \text{for all } t > 0$$

and the initial condition

$$u(x,0) = u_0(x)$$

Lets try to find a (steady state) particular solution $\varphi(x)$ that satisfies the equation and the non-homogeneous boundary conditions. In other words, we want

$$0 = \alpha^2 \varphi''(x) + f(x)$$

with

$$\varphi(0) = A, \varphi(a) = B \quad \text{for all } t > 0$$

The equation for φ can be solved by simply integrating twice. This yields

$$\begin{aligned} \varphi(x) &= -\alpha^{-2} \int_0^x \int_0^s f(r) dr ds + bx + c \\ &= -\alpha^{-2} \int_0^x (x-r) f(r) dr + bx + c \end{aligned}$$

where b and c are arbitrary constants. These constants can be adjusted to satisfy the boundary conditions. We want

$$\varphi(0) = c = A$$

and

$$\varphi(a) = -\alpha^{-2} \int_0^a (a-r) f(r) dr + ba + c = B$$

so we set $c = A$ and $b = (B - A + \alpha^{-2} \int_0^a (a-r) f(r) dr)/a$.

Now, since $f(x) = -\alpha^2 \varphi''(x)$ we find that $v(x,t) = u(x,t) - \varphi(x)$ solves the heat equation without the non-homogeneous term:

$$\begin{aligned} \frac{\partial v(x,t)}{\partial t} &= \frac{\partial u(x,t)}{\partial t} \\ &= \alpha^2 \frac{\partial^2 u(x,t)}{\partial x^2} + f(x) \\ &= \alpha^2 \frac{\partial^2 u(x,t)}{\partial x^2} - \alpha^2 \frac{\partial^2 \varphi(x)}{\partial x^2} \\ &= \alpha^2 \frac{\partial^2 v(x,t)}{\partial x^2} \end{aligned}$$

Also, $v(x,t)$ satisfies the boundary condition

$$v(0,t) = u(0,t) - \varphi(0) = A - A = 0, \quad v(a,t) = u(a,t) - \varphi(b) = B - B = 0 \quad \text{for all } t > 0$$

and the initial condition

$$v(x,0) = u_0(x) - \varphi(x)$$

Thus we can find $v(x,t)$ in the form of a sine series, and then obtain $u(x,t) = v(x,t) + \varphi(x)$.

Problem 3.7: Solve

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + x,$$

with the non-homogeneous boundary condition

$$u(0,t) = 0, \quad u(1,t) = 1 \quad \text{for all } t > 0$$

and the initial condition

$$u(x,0) = 0$$

Non-homogeneous term $f(x,t)$ in the equation

What can we do when the non-homogeneous term does depend on t as well? Instead of removing the inhomogeneity in the equation and the boundary condition all at once we do them one at a time.

Lets consider the equation

$$\frac{\partial u(x,t)}{\partial t} = \alpha^2 \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t),$$

with the non-homogeneous boundary condition

$$u(0,t) = A, \quad u(a,t) = B \quad \text{for all } t > 0$$

and the initial condition

$$u(x,0) = u_0(x)$$

First we will find a particular solution $\varphi(x,t)$ to the equation. Lets try to find such a solution as a sine series. Then this particular solution will satisfy zero boundary conditions so adding or subtracting it will have no effect on the boundary conditions.

First we expand $f(x,t)$ in a sine series. We have

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t) \sin(n\pi x/a)$$

where

$$f_n(t) = \frac{2}{a} \int_0^a \sin(n\pi x/a) f(x,t) dx$$

Next we write

$$\varphi(x,t) = \sum_{n=1}^{\infty} \gamma_n(t) \sin(n\pi x/a).$$

If we plug φ into the equation we obtain

$$\sum_{n=1}^{\infty} \gamma_n'(t) \sin(n\pi x/a) = \sum_{n=1}^{\infty} -(n\pi/a)^2 \gamma_n(t) \sin(n\pi x/a) + \sum_{n=1}^{\infty} f_n(t) \sin(n\pi x/a)$$

so the equation will be satisfied if

$$\gamma_n'(t) = -(n\pi/a)^2 \gamma_n(t) + f_n(t)$$

for every n. This holds if

$$\gamma_n(t) = e^{-(n\pi/a)^2 t} \int_0^t e^{(n\pi/a)^2 \tau} f_n(\tau) d\tau + g_n e^{-(n\pi/a)^2 t}$$

for any constant g_n. Any choices of g_n will give us a particular solution. Since there is no apparent reason to prefer one over the other, we will simply choose to set $g_n = 0$ for every n. Then

$$\varphi(x,t) = \sum_{n=1}^{\infty} e^{-(n\pi/a)^2 t} \int_0^t e^{(n\pi/a)^2 \tau} f_n(\tau) d\tau \sin(n\pi x/a)$$

is a particular solution of the equation with the non-homogeneous term.

Now set $v(x,t) = u(x,t) - \varphi(x,t)$. Then $v(x,t)$ satisfies the heat equation

$$\frac{\partial v(x,t)}{\partial t} = \alpha^2 \frac{\partial^2 v(x,t)}{\partial x^2},$$

with the non-homogeneous boundary condition as u

$$v(0,t) = A, \quad v(a,t) = B \quad \text{for all } t > 0$$

and the initial condition

$$v(x,0) = u_0(x) - \varphi(x,0)$$

But this is a problem that we have already seen how to solve. We must make another subtraction, and subtract a steady state solution $\psi(x)$ from $v(x,t)$. Then $w(x,t) = v(x,t) - \psi(x)$ will solve the heat equation with homogeneous boundary conditions. We can find $w(x,t)$ as a sine series, and finally obtain $u(x,t) = w(x,t) + \psi(x) + \varphi(x,t)$.

Problem 3.8: Solve

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + t,$$

with the non-homogeneous boundary condition

$$u(0,t) = 0, \quad u(1,t) = 1 \quad \text{for all } t > 0$$

and the initial condition

$$u(x,0) = 0$$

Problem 3.9: How could you solve

$$\frac{\partial u(x,t)}{\partial t} = \alpha^2 \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t),$$

with the boundary condition

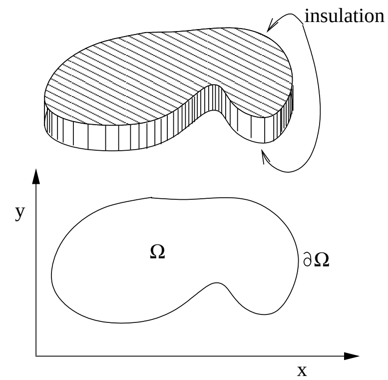
$$\frac{\partial u(0,t)}{\partial x} = A, \quad \frac{\partial u(a,t)}{\partial x} = B \quad \text{for all } t > 0$$

and the initial condition

$$u(x,0) = u_0(x)$$

The heat equation in two space dimensions

We will now study a heat flow problem in two space dimensions. If a thin plate is insulated on the top and the bottom, then, to a good approximation, the temperature function is constant across the plate from top to bottom, and only depends on two variables, say x and y. These variables range within a domain Ω determined by the shape of the plate.



The temperature function $u(x, y, t)$ is now a function of three variables—two space variables and time. It satisfies the heat equation

$$\frac{\partial u}{\partial t} = \alpha^2 \Delta u = \alpha^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x^2} \right)$$

together with boundary conditions and an initial condition.

As we did in the case of one space dimension, we first consider homogeneous boundary conditions—Dirichlet and Neumann. In two space dimensions, the boundary of Ω, denoted $\partial\Omega$, is a curve in the plane. The temperature function $u(x, y, t)$ satisfies Dirichlet boundary conditions if

$$u(x, y, t) = 0 \quad \text{for } (x, y) \in \partial\Omega$$

This means that the temperature is kept at a constant $0°$ all around the boundary.

Neumann boundary conditions are satisfied if the the directional derivative of u in the direction normal (i.e., at right angles) to the boundary is zero. In other words

$$\frac{\partial u(x, y, t)}{\partial \nu} = 0 \quad \text{for } (x, y) \in \partial\Omega$$

where $\frac{\partial u(x,y,t)}{\partial \nu}$ denotes the normal derivative. If Neumann boundary condition hold, then there is no heat flux across the boundary, i.e., the boundary of the plate is insulated.

The initial condition is given by

$$u(x, y, 0) = u_0(x, y) \quad \text{for } (x, y) \in \Omega$$

where $u_0(x, y)$ is the given initial temperature distribution.

Solving the 2D problem with homogeneous boundary conditions: general case

Lets start with Dirichlet boundary conditions. The approach we take is analogous to the one space dimensional case: we expand the solution in a basis of eigenfunctions.

So to start with, we find all the eigenvalues and eigenfunctions of Δ with Dirichlet boundary conditions. In other words we need to find all possible solutions λ and $\phi(x, y)$ to the equation

$$-\Delta\phi(x, y) = \lambda\phi(x, y)$$

satisfying

$$\phi(x, y) = 0 \quad \text{for } (x, y) \in \partial\Omega$$

In the situations where we can actually calculate these, the solutions are indexed by two integers, so we will denote them by $\lambda_{n,m}$ and $\phi_{n,m}(x, y)$.

There is a theorem that states that all the eigenvalues $\lambda_{n,m}$ are real (in fact non-negative) and that the eigenfunctions $\phi_{n,m}(x, y)$ form an orthogonal basis. The orthogonality condition is

$$\langle\phi_{n,m}, \phi_{n',m'}\rangle = \int\int_\Omega \phi_{n,m}(x, y)\phi_{n',m'}(x, y) = 0 \quad \text{unless } n = n' \text{ and } m = m'$$

Saying the functions form a basis means that we can expand "any" function $\psi(x, y)$ in a series

$$\psi(x, y) = \sum_n \sum_m b_{n,m}\phi_{n,m}(x, y)$$

There are, as in one dimension, technical questions about when the left side converges and is equal to the right side. I will just sweep these under the rug. Certainly, the sum on the right always satisfies Dirichlet boundary conditions, since each $\phi_{n,m}$ does. The coefficients are found by using the orthogonality relations

$$\langle\phi_{n',m'}, \psi\rangle = \sum_n \sum_m b_{n,m}\langle\phi_{n',m'}, \phi_{n,m}\rangle = b_{n',m'}\langle\phi_{n',m'}, \phi_{n',m'}\rangle$$

so that

$$b_{n,m} = \langle\phi_{n,m}, \psi\rangle / \langle\phi_{n,m}, \phi_{n,m}\rangle$$

Now we expand our unknown solution as

$$u(x, y, t) = \sum_n \sum_m \beta_{m,n}(t)\phi_{n,m}(x, y)$$

for unknown coefficients $\beta_{n,m}(t)$. This expression automatically satisfies the right boundary conditions. Plugging this into the equation, and using the fact that the functions $\phi_{n,m}$ are eigenfunctions we find

$$\sum_n \sum_m \beta'_{m,n}(t)\phi_{n,m}(x, y) = \alpha^2 \sum_n \sum_m \beta_{m,n}(t)\Delta\phi_{n,m}(x, y) = \alpha^2 \sum_n \sum_m \beta_{m,n}(t)(-\lambda_{n,m})\phi_{n,m}(x, y)$$

This means that to satisfy the equation, we must have

$$\beta'_{m,n}(t) = -\alpha^2\lambda_{n,m}\beta_{m,n}(t)$$

or,

$$\beta_{n,m}(t) = b_{n,m}e^{-\alpha^2\lambda_{n,m}t}$$

for some constants $b_{n,m}$. These constants are determined by the initial condition

$$u(x,y,0) = \sum_n \sum_m b_{m,n}\phi_{n,m}(x,y) = u_0(x,y)$$

so that

$$\begin{aligned} b_{n,m} &= \langle \phi_{n,m}, u_0 \rangle / \langle \phi_{n,m}, \phi_{n,m} \rangle \\ &= \int\int_\Omega \phi_{n,m}(x,y)u_0(x,y)dxdy / \int\int_\Omega \phi_{n,m}^2(x,y)dxdy \end{aligned}$$

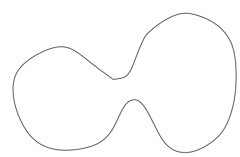
The final form of the solution is then

$$u(x,y,t) = \sum_n \sum_m b_{m,n} e^{-\alpha^2 \lambda_{n,m} t} \phi_{n,m}(x,y)$$

The case of Neumann boundary conditions is completely analogous, except that the eigenfunctions are now required to satisfied Neumann boundary conditions. We end up with a new set of eigenvalues $\lambda_{n,m}$ and eigenfunctions $\phi_{n,m}(x,y)$ and expand the solution in terms of these.

Unfortunately, unless the domain is something very special like a rectangle or a circle, it is impossible to calculate the eigenvalues $\lambda_{n,m}$ and eigenfunctions $\phi_{n,m}(x,y)$ explicitly. However, many mathematicians have worked on understanding the relationship between the shape of Ω and the eigenvalues.

For example, if Ω has a narrow neck like this



then the first non-zero eigenvalue will be very small. Why? Well, if the $\lambda_{n,m}$'s are all large then all the terms $e^{-\alpha^2 \lambda_{n,m} t}$ in the solution of the heat equation will tend to zero very quickly, and the equilibrium heat distribution will be achieved in a very short time. However, thats not possible if the heat has to diffuse through a narrow neck.

Another interesting problem, which was only solved in the last ten years is the following: Do the Dirichlet eigenvalues $\lambda_{n,m}$ determine the domain, or is it possible for two different domains to have exactly the same Dirichlet eigenvalues. Surprisingly, the answer is: it is possible!

Homogeneous boundary conditions for a rectangle

If the domain Ω is a rectangle then we can find the Dirichlet and Neumann eigenvalues and eigenfunctions explicitly. We have already encountered them when doing double Fourier expansions. Suppose the rectangle contains all points (x,y) with $0 \le x \le a$ and $0 \le y \le b$.

For Dirichlet boundary conditions,

$$\phi_{n,m}(x,y) = \sin(n\pi x/a)\sin(n\pi y/b)$$

for $n = 1, 2, 3, \ldots$ and $m = 1, 2, 3, \ldots$.

Lets check that the Dirichlet boundary conditions are satisfied by each of these functions. The boundary consists of four line segments, $\{(x, 0) : 0 \le x \le a\}$,$\{(a, y) : 0 \le y \le b\}$,$\{(x, b) : 0 \le x \le a\}$ and $\{(0, y) : 0 \le y \le b\}$. It is easy to verify that $\phi_{n,m}(x, y) = 0$ along each of these segments.

Next, lets verify that these functions are eigenfunctions of the Laplace operator. We calculate

$$\begin{aligned}
-\Delta\phi_{n,m}(x, y) &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x^2}\right)\phi_{n,m}(x, y) \\
&= \left(\left(\frac{n\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right)\phi_{n,m}(x, y)
\end{aligned}$$

Thus $\phi_{n,m}(x, y)$ is an eigenfunction with eigenvalue

$$\lambda_{n,m} = \left(\frac{n\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \tag{3.1}$$

What we won't be able to show here is that this is a complete listing of all the eigenfunctions and eigenvalues. But, in fact, it is a complete list. So by the formula in the previous section, the solution to heat equation with Dirichlet boundary conditions and initial condition u_0 is

$$u(x, y, t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_{m,n}e^{-\alpha^2\lambda_{n,m}t}\sin(n\pi x/a)\sin(n\pi y/b)$$

where

$$b_{n,m} = \frac{4}{ab}\int_0^a\int_0^b \sin(n\pi x/a)\sin(n\pi y/b)u_0(x, y)dxdy$$

Similarly, for Neumann boundary conditions, the eigenfunctions are

$$\phi_{0,0} = \frac{1}{4}$$

$$\phi_{n,0} = \frac{1}{2}\cos(n\pi x/a)$$

$$\phi_{0,m} = \frac{1}{2}\cos(m\pi y/b)$$

$$\phi_{n,m}(x, y) = \cos(n\pi x/a)\cos(n\pi y/b)$$

for $n = 1, 2, 3, \ldots$ and $m = 1, 2, 3, \ldots$. and the eigenvalues $\lambda_{n,m}$ are given by the same formula () above.

Thus the solution to heat equation with insulated sides and initial condition u_0 is

$$\begin{aligned}
u(x, y, t) = &\frac{a_{0,0}}{4} \\
&+ \sum_{n=1}^{\infty}\frac{a_{n,0}}{2}e^{-\alpha^2\lambda_{n,0}t}\cos(n\pi x/a) \\
&+ \sum_{m=1}^{\infty}\frac{a_{0,m}}{2}e^{-\alpha^2\lambda_{0,m}t}\cos(m\pi y/b) \\
&+ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{m,n}e^{-\alpha^2\lambda_{n,m}t}\cos(n\pi x/a)\cos(n\pi y/b)
\end{aligned}$$

where

$$a_{n,m} = \frac{4}{ab}\int_0^a\int_0^b \cos(n\pi x/a)\cos(n\pi y/b)u_0(x, y)dxdy$$

Example

Lets solve the heat equation

$$\frac{\partial u}{\partial t} = \Delta u$$

on the square $0 \le x \le 1$ and $0 \le y \le 1$ with Dirichlet boundary conditions and initial conditions

$$u(x,y,0) = \begin{cases} 0 & \text{if } y > x \\ 1 & \text{if } x \ge y \end{cases}$$

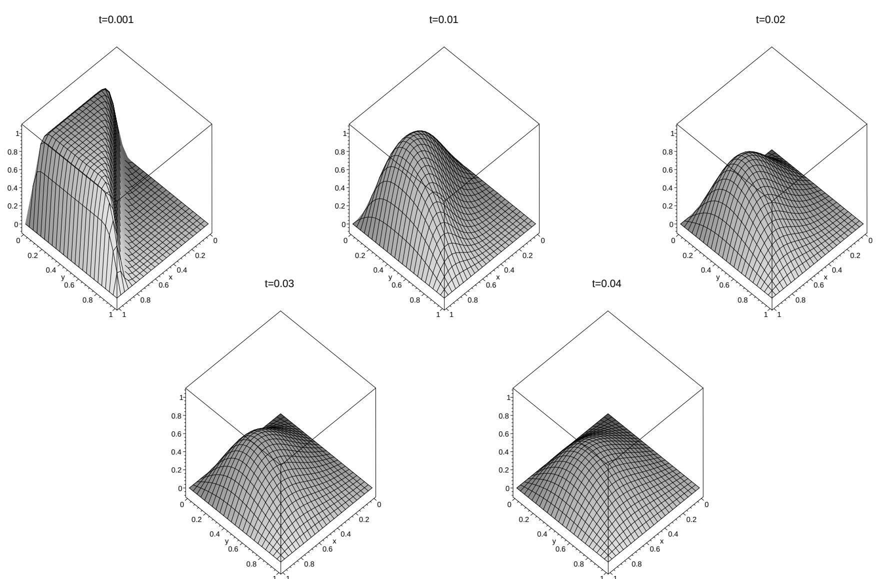
The solution is given by

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_{m,n} e^{-\lambda_{n,m} t} \sin(n\pi x)\sin(n\pi y)$$

where the $b_{n,m}$'s are the coefficients of $u(x,y,0)$ when expanded in a double sine series. This is a calculation that was done in a homework problem. The answer is

$$b_{n,m} = 4\frac{(-1)^n}{nm\pi^2}((-1)^m - 1) + \frac{4}{n\pi}\begin{cases} \frac{m(1-(-1)^{n+m})}{\pi(m^2-n^2)} & \text{if } n \ne m \\ 0 & \text{if } n = m \end{cases}$$

Here are some plots of the function $u(x,y,t)$ as t increases. Notice how, with Dirichlet boundary conditions, the heat just drains away.



Problem 3.10: Solve the heat equation

$$\frac{\partial u}{\partial t} = \Delta u$$

on the square $0 \leq x \leq 1$ and $0 \leq y \leq 1$ with Neumann boundary conditions and initial conditions

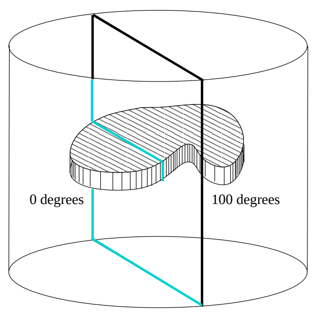
$$u(x,y,0) = \begin{cases} 0 & \text{if } y > x \\ 1 & \text{if } x \geq y \end{cases}$$

Problem 3.11: Describe how to solve the heat flow problem for a plate that is insulated on the top and bottom, and on two opposite sides, while the other two opposite sides are kept at zero temperature. Assume the initial temperature distribution is $u_0(x,y)$.

2D heat equations: non-homogeneous boundary conditions

How can we compute the temperature flow in a two dimensional region Ω when the temperatures around the boundary $\partial\Omega$ are held at fixed values other than zero?

For example we might be interested in the following experimental setup:



An insulating wall is placed in a beaker. One side is filled with ice water and the other side with boiling water. The metal plate (with top and bottom insulated) is inserted as shown. Then, the part of the boundary in the ice water will be kept at a constant temperature of $0°$ while the part in the boiling water will be kept at $100°$.

To solve this problem, we must solve the heat equation

$$\frac{\partial u}{\partial t} = \alpha^2 \Delta u = \alpha^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x^2} \right)$$

with boundary condition

$$u(x,y,t) = f(x,y) \quad \text{for } (x,y) \in \partial\Omega$$

and initial condition

$$u(x,y,0) = u_0(x,y)$$

Here the function $f(x,y)$ is defined for values of (x,y) along the boundary curve $\partial\Omega$ surrounding Ω. The values of $f(x,y)$ are the fixed temperatures. In the example above, $f(x,y) = 0$ for (x,y) in the part of the boundary lying in the ice water, while $f(x,y) = 100$ for (x,y) in the part of the boundary lying in the boiling water.

We adopt the same strategy that we used in one space dimension. We first look for a steady state solution that satisfies the equation and boundary condition. Thus the steady state solution is a function $\varphi(x,y)$ defined for $(x,y) \in \Omega$ that satisfies the equation

$$\Delta\varphi = \frac{\partial \varphi}{\partial x^2} + \frac{\partial \varphi}{\partial y^2} = 0$$

and the boundary condition

$$\varphi(x,y) = f(x,y) \quad \text{for } (x,y) \in \partial\Omega$$

This is Laplace's equation, which we will study next. At the moment it is not at all clear how to solve this equation. There is one special case where it is easy to solve—when $f(x,y) = C$ is constant. Then the constant function $\varphi(x,y) = C$ is the solution. This makes sense: if we hold the whole boundary at a steady C degrees, then eventually the whole plate should end up at that temperature.

Suppose now that we have obtained the solution $\varphi(x,y)$ to Laplaces equation with boundary condtion. We define $v(x,y,t) = u(x,y,t) - \varphi(x,y)$. Then $v(x,y,t)$ will satisfy the heat equation with homogeneous boundary condition

$$v(x,y,t) = 0 \quad \text{for } (x,y) \in \partial\Omega$$

and initial condition

$$v(x,y,0) = u_0(x,y) - \varphi(x,y)$$

Using the method of the previous sections, we can solve for $v(x,y,t)$ in the form of an eigenfunction expansion. Then we let

$$u(x,y,t) = v(x,y,t) + \varphi(x,y)$$

to obtain the final solution.

At this point you can go back to the introductory lecture and understand in detail how the solution of the heat equation presented there was obtained.

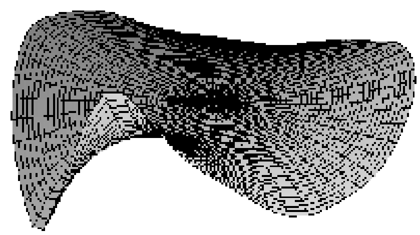
Problem 3.12: Go back to the introductory lecture and understand in detail how the solution of the heat equation presented there was obtained.

Problem 3.13: Consider a rectangular metal plate, insulated on the top and bottom, of size 10 cm $\times$ 20 cm. Initially, the plate is at a constant temperature of 0°. Then, it is placed in a tub of water at 50°. Find the temperature at all later times. Assume $\alpha^2 = 1$.

Laplace's equation

We have already seen that solutions to Laplace's equation describe heat distributions at equilibrium. Here are some other examples where solutions to Laplace's equation describe equilibrium configurations.

Suppose we take rubber membrane and attach it to a stiff wire loop. What will be the shape of the membrane?



In this example, let $\varphi(x, y)$ denote the height of the membrane above the x–y plane. Let Ω denote the set of points lying under the membrane on the x–y plane. Then $\varphi(x, y)$ is the solution of Laplace's equation

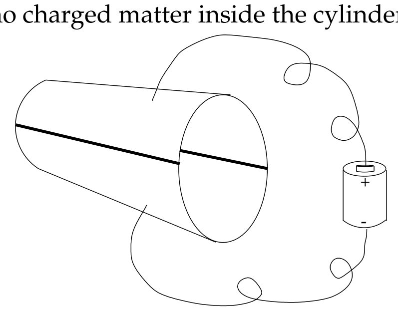
$$\Delta\varphi(x, y) = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0$$

for $(x, y) \in \Omega$, with a boundary condition determined by the shape of the wire. If $f(x, y)$ for $(x, y) \in \partial\Omega$ denotes the height of wire above the x–y plane, then we require

$$\varphi(x, y) = f(x, y) \quad \text{for } (x, y) \in \partial\Omega$$

In this example, the shape of the membrane is the equilibrium configuration of a vibrating drum head.

Another example where Laplace's equation plays an important role is electrostatics. Suppose we make a cylinder out of two half round pieces of metal fastened together with a thin insulating layer, and attach each side to opposite ends of a battery. If there is no charged matter inside the cylinder, what will be the electric field inside?



In this example, the electrostatic potential φ is really a function of three space variables. The equation for the electrostatic potential is

$$\Delta\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = \rho(x, y, z)$$

where $\rho(x, y, z)$ is the charge density. In our example the charge density is zero. Also, if the cylinder is infinitely long, we may assume that φ only depends on two variables and is constant along the cylinder. Then $\varphi(x, y)$ satisfies Laplace's equation in two variables. The boundary condition is the requirement that the potential be constant on each metal half. The electric field (which is proportional to the force felt by a charged particle) is then given by the gradient $\mathbf{E} = \nabla\varphi$.

Laplace's equation in a rectangle

Suppose Ω is a rectangle $0 \le x \le a$ and $0 \le b \le b$ We wish to solve

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0$$

for $0 \le x \le a$ and $0 \le b \le b$ with

$$\begin{aligned}
\varphi(x,0) &= f_1(x) \quad \text{for } 0 \le x \le a\\
\varphi(a,y) &= f_2(y) \quad \text{for } 0 \le y \le b\\
\varphi(x,b) &= f_3(x) \quad \text{for } 0 \le x \le a\\
\varphi(0,y) &= f_4(y) \quad \text{for } 0 \le y \le b
\end{aligned}$$

To solve this problem, we break it into four easier ones, where the boundary condition is zero on three of the four boundary pieces. In other words, we let φ_1 be the solution of Laplace's equation $\Delta\varphi_1 = 0$ with

$$\begin{aligned}
\varphi(x,0) &= f_1(x) \quad \text{for } 0 \le x \le a\\
\varphi(a,y) &= 0 \quad \text{for } 0 \le y \le b\\
\varphi(x,b) &= 0 \quad \text{for } 0 \le x \le a\\
\varphi(0,y) &= 0 \quad \text{for } 0 \le y \le b
\end{aligned} \tag{4.1}$$

and similarly for φ_2, φ_3 andφ_4. Then when we add this solutions up, the result $\varphi = \varphi_1 + \varphi_2 + \varphi_3 + \varphi_4$ will satisfy Laplace's equation, as well as the correct boundary condition on all four sides.

How can we solve the easier problem—Laplace's equation with boundary condition (4.1)? We expand the solution $\varphi(x,y)$ into a sine series in x. Thus

$$\varphi(x,y) = \sum_{n=1}^{\infty} \phi_n(y)\sin(n\pi x/a)$$

Then φ will automatically vanish at $x = 0$ and $x = a$, so two of the four boundary conditions are met. The coefficients in this expansion are undetermined functions of y.

If we substitute this expansion into Laplace's equation, we get

$$\Delta\varphi = \sum_{n=1}^{\infty} \phi_n''(y)\sin(n\pi x/a) - (n\pi/a)^2\phi_n(y)\sin(n\pi x/a)$$

so the equation holds if for every n

$$\phi_n''(y) = (n\pi/a)^2\phi_n(y)$$

or, if

$$\phi_n(y) = a_n e^{n\pi y/a} + b_n e^{-n\pi y/a}$$

Now we try to satisfy the other two boundary conditions. When $y = b$ then we want ϕ_n to be zero

$$\phi_n(b) = a_n e^{n\pi b/a} + b_n e^{-n\pi b/a}$$

set

$$b_n = -e^{2n\pi b/a}a_n$$

Then $\phi_n(b)$ will be zero for each n, and the third boundary condition is satisfies. It remains to satisfy

$$\begin{aligned}
\varphi(x,0) &= \sum_{n=1}^{\infty}(a_n + b_n)\sin(n\pi x/a)\\
&= \sum_{n=1}^{\infty} a_n(1 - e^{2n\pi b/a})\sin(n\pi x/a)\\
&= f_1(x)
\end{aligned}$$

Thus $a_n(1-e^{2n\pi b/a})$ are the coefficients in the sine expansion of $f_1(x)$. In other words

$$a_n(1-e^{2n\pi b/a}) = \frac{2}{a}\int_0^a \sin(n\pi x/a) f_1(x)dx$$

This determines the a_n's and so the solution to the problem with boundary conditions () is complete.

Example

Lets solve Laplace's equation on the unit square $0 \le x \le 1$ and $0 \le y \le 1$ with boundary condition

$$\begin{aligned}
\varphi(x,0) &= \begin{cases} x & \text{if } 0 \le x \le 1/2 \\ 1-x & \text{if } 1/2 \le x \le 1 \end{cases} \\
\varphi(1,y) &= 0 \quad \text{for } 0 \le y \le 1 \\
\varphi(x,1) &= 0 \quad \text{for } 0 \le x \le 1 \\
\varphi(0,y) &= 0 \quad \text{for } 0 \le y \le 1
\end{aligned}$$

We have

$$\begin{aligned}
a_n &= \frac{2}{1-e^{2n\pi}}\left[\int_0^{1/2} x\sin(n\pi x)dx + \int_{1/2}^1 (1-x)\sin(n\pi x)dx\right] \\
&= \frac{2}{1-e^{2n\pi}}\frac{2\sin(n\pi/2)}{n^2\pi^2} \\
&= \frac{4\sin(n\pi/2)}{(1-e^{2n\pi})n^2\pi^2}
\end{aligned}$$

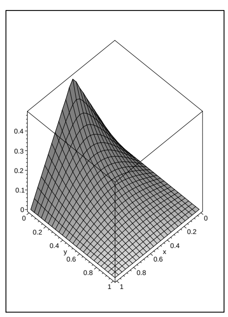
and

$$b_n = -e^{2n\pi b/a}a_n = \frac{4\sin(n\pi/2)}{(1-e^{-2n\pi})n^2\pi^2}$$

With these values of a_n and b_n

$$\varphi(x,y) = \sum_{n=1}^{\infty}(a_n e^{n\pi y} + b_n e^{-n\pi y})\sin(n\pi x)$$

Here is a picture of the solution



Problem 4.1: Solve Laplace's equation on the unit square $0 \le x \le 1$ and $0 \le y \le 1$ with boundary condition

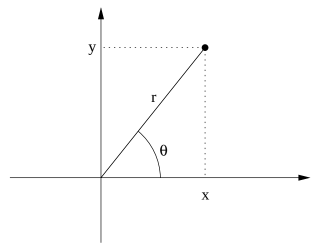
$$\begin{aligned}
\varphi(x,0) &= x \quad \text{for } 0 \le x \le 1 \\
\varphi(1,y) &= 1-y \quad \text{for } 0 \le y \le 1 \\
\varphi(x,1) &= 0 \quad \text{for } 0 \le x \le 1 \\
\varphi(0,y) &= 0 \quad \text{for } 0 \le y \le 1
\end{aligned}$$

Polar co-ordinates

One of the most powerful ways of simplifying (and also of complicating!) a PDE is to make a change of co-ordinates (or independent variables).

Probably the most common co-ordinates, after the usual cartesian co-ordinates we have used until now, are polar co-ordinates.

To every point (x, y) in the plane we associate two numbers r and θ. The number r is the distance of (x, y) from the origin, and θ is the angle that the line through the origin and (x, y) makes with the x axis.



Given r and θ we can solve for x and y. This defines x and y as functions of r and θ.

$$\begin{aligned} x &= x(r,\theta) &= r\cos(\theta) \\ y &= y(r,\theta) &= r\sin(\theta) \end{aligned}$$

Notice that if we change θ by a multiple of 2π, the corresponding x and y values don't change, since we have simply gone around the circle some number of times and ended up at the same spot.

Conversely, given x and y, we can solve for r and θ. Actually, we can't quite solve for θ, because there are infinitely many θ values, all differing by a multiple of 2π, that correspond to the same x and y. However, if we insist that, for example, $-\pi < \theta \le \pi$ then θ is uniquely determined and we have

$$\begin{aligned} r &= r(x,y) &= \sqrt{x^2+y^2} \\ \theta &= \theta(x,y) &= \operatorname{atan}(x,y) \quad = \begin{cases} \arctan(y/x) & \text{for } x \ge 0 \\ \arctan(y/x)+\pi & \text{for } x < 0 \text{ and } y \ge 0 \\ \arctan(y/x)-\pi & \text{for } x < 0 \text{ and } y < 0 \end{cases} \end{aligned}$$

Given a function $\phi(x, y)$ we can define a new function $\Phi(r, \theta)$ by

$$\Phi(r,\theta) = \phi(x(r,\theta), y(r\theta))$$

In other words, if r, θ and x, y are the polar and cartesian co-ordinates of the same point, then $\Phi(r, \theta) = \phi(x, y)$. (Warning: often you will see $\Phi(r, \theta)$ written as $\phi(r, \theta)$, which can be confusing at first.) Notice that $\Phi(r, \theta+2\pi k) = \Phi(r, \theta)$ for $k \in \mathbb{Z}$. This means that $\Phi(r, \theta)$ is a periodic function of θ with period 2π. The relation between ϕ and Φ can also be written in the form

$$\phi(x,y) = \Phi(r(x,y), \theta(x,y)).$$

Now suppose that $\phi(x, y)$ solves some differential equation (like Laplace's equation.) Then $\Phi(r, \theta)$ will also solve some differential equation. To determine which one, we must apply the chain rule for two variables.

Recall that

$$\begin{aligned} \frac{\partial\Phi}{\partial r} &= \frac{\partial\phi}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial r} = \cos(\theta)\frac{\partial\phi}{\partial x} + \sin(\theta)\frac{\partial\phi}{\partial y} \\ \frac{\partial\Phi}{\partial\theta} &= \frac{\partial\phi}{\partial x}\frac{\partial x}{\partial\theta} + \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial\theta} = -r\sin(\theta)\frac{\partial\phi}{\partial x} + r\cos(\theta)\frac{\partial\phi}{\partial y} \end{aligned}$$

Inverting this system of linear equations, we find

$$\begin{aligned}
\frac{\partial \phi}{\partial x} &= \cos(\theta)\frac{\partial \Phi}{\partial r} - \frac{\sin(\theta)}{r}\frac{\partial \Phi}{\partial \theta} \\
\frac{\partial \phi}{\partial y} &= \sin(\theta)\frac{\partial \Phi}{\partial r} + \frac{\cos(\theta)}{r}\frac{\partial \Phi}{\partial \theta}
\end{aligned}$$

Now we compute the second derivatives. We have to use the formula above twice, and the product rule.

$$\begin{aligned}
\frac{\partial^2 \phi}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right) \\
&= \frac{\partial}{\partial x}\left(\cos(\theta)\frac{\partial \Phi}{\partial r} - \frac{\sin(\theta)}{r}\frac{\partial \Phi}{\partial \theta}\right) \\
&= \cos(\theta)\frac{\partial}{\partial r}\left(\cos(\theta)\frac{\partial \Phi}{\partial r} - \frac{\sin(\theta)}{r}\frac{\partial \Phi}{\partial \theta}\right) - \frac{\sin(\theta)}{r}\frac{\partial}{\partial \theta}\left(\cos(\theta)\frac{\partial \Phi}{\partial r} - \frac{\sin(\theta)}{r}\frac{\partial \Phi}{\partial \theta}\right) \\
&= \cos^2(\theta)\frac{\partial^2 \Phi}{\partial r^2} + \frac{\cos(\theta)\sin(\theta)}{r^2}\frac{\partial \Phi}{\partial \theta} - \frac{\cos(\theta)\sin(\theta)}{r}\frac{\partial^2 \Phi}{\partial r \partial \theta} + \frac{\sin^2(\theta)}{r}\frac{\partial \Phi}{\partial r} \\
&\quad - \frac{\sin(\theta)\cos(\theta)}{r}\frac{\partial^2 \Phi}{\partial \theta \partial r} + \frac{\sin(\theta)\cos(\theta)}{r^2}\frac{\partial \Phi}{\partial \theta} + \frac{\sin^2(\theta)}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2}
\end{aligned}$$

A similar calculation for the y derivatives yields

$$\begin{aligned}
\frac{\partial^2 \phi}{\partial y^2} &= \sin^2(\theta)\frac{\partial^2 \Phi}{\partial r^2} + \frac{\sin(\theta)\cos(\theta)}{r}\frac{\partial^2 \Phi}{\partial r \partial \theta} - \frac{\sin(\theta)\cos(\theta)}{r^2}\frac{\partial \Phi}{\partial \theta} + \frac{\cos^2(\theta)}{r}\frac{\partial \Phi}{\partial r} \\
&\quad + \frac{\cos(\theta)\sin(\theta)}{r}\frac{\partial^2 \Phi}{\partial \theta \partial r} - \frac{\cos(\theta)\sin(\theta)}{r^2}\frac{\partial \Phi}{\partial \theta} + \frac{\cos^2(\theta)}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2}
\end{aligned}$$

Adding these equations, and using the identity $\sin^2(\theta) + \cos^2(\theta) = 1$ we find

$$\begin{aligned}
\Delta \phi &= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \\
&= \frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2}
\end{aligned}$$

Therefore, if ϕ solves Laplace's equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

then Φ solves

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} = 0$$

The second equation is called Laplace's equation in polar co-ordinates. Although these equations look different (and *are* different) they are equivalent. If we can solve one, then we can easily obtain a solution to the other by using the formulas relating ϕ and Φ.

Laplace's equation on a disk

Polar co-ordinates are perfectly suited for solving Laplace's equation on a disk. In polar co-ordinates, a disk is given by $\{(r, \theta) : r < R\}$, and a boundary condition can be specified with a function $f(\theta)$ depending on θ only. Thus we must solve

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} = 0$$

subject to

$$\Phi(R,\theta) = f(\theta)$$

To do this we expand $\Phi(r,\theta)$ in a Fourier series in the θ variable. Since $\Phi(r,\theta)$ is a periodic function of θ with period $L = 2\pi$ we use the standard Fourier series. We may use either the complex exponential form or the form with sines and cosines. Lets use the complex exponentials $e^{2\pi n\theta/L} = e^{in\theta}$. Notice that these functions are eigenfunctions for $\frac{\partial^2}{\partial\theta^2}$.

$$\Phi(r,\theta) = \sum_{n=-\infty}^{\infty} \gamma_n(r)e^{in\theta}$$

Subsituting this into the equation, we obtain

$$\begin{aligned}\frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} &= \sum_{n=-\infty}^{\infty} \gamma_n''(r)e^{in\theta} + \frac{\gamma_n'(r)}{r}e^{in\theta} + \frac{\gamma_n(r)}{r^2}\frac{\partial^2}{\partial\theta^2}e^{in\theta}\\ &= \sum_{n=-\infty}^{\infty}\left(\gamma_n''(r) + \frac{\gamma_n'(r)}{r} - \frac{n^2}{r^2}\gamma_n(r)\right)e^{in\theta}\\ &= 0\end{aligned}$$

So the equation holds if for every n

$$\gamma_n''(r) + \frac{\gamma_n'(r)}{r} - \frac{n^2}{r^2}\gamma_n(r) = 0$$

This is an Euler equation whose general solution is $c_n r^n + d_n r^{-n}$ when $n \neq 0$ and $c_n + d_n\ln(r)$ when $n = 0$. Notice that in each case, one of the solutions blows up at the origin. Since we are looking for a solution that is continuous at the origin, we throw away the singular solution. Thus we have $\gamma_n(r) = c_n r^{|n|}$ when $n \neq 0$, and c_0 when $n = 0$, and

$$\Phi(r,\theta) = \sum_{n=-\infty}^{\infty} c_n r^{|n|}e^{in\theta}$$

To satisfy the initial condition we require that

$$\Phi(R,\theta) = \sum_{n=-\infty}^{\infty} c_n a^{|n|}e^{in\theta} = f(\theta)$$

In other words, $c_n R^{|n|}$ are the Fourier coeficients of f. Thus

$$c_n = \frac{1}{2\pi R^{|n|}}\int_0^{2\pi} e^{-in\theta}f(\theta)d\theta$$

This determines the c_n's and hence the solution. Recall that in this integral we may equally well integrate from $-\pi$ to π (or over any other interval of length 2π).

If we use sines and cosines to expand Φ, we end up with

$$\Phi(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos(n\theta) + b_n\sin(n\theta))r^{|n|}$$

with

$$a_n = \frac{1}{\pi R^n}\int_0^{2\pi}\cos(n\theta)f(\theta)d\theta$$

and

$$b_n = \frac{1}{\pi R^n}\int_0^{2\pi}\sin(n\theta)f(\theta)d\theta$$

Example

As an example, lets solve the problem of finding the electrostatic potential inside the infinite cylinder. Suppose the radius of the cylinder is 1, and that the bottom half is kept at voltage 0 while the top half is kept at voltage 1. We choose the co-ordinates so that $-\pi < \theta \le 0$ corresponds to the bottom half of the cylinder and $0 \le \theta \le \pi$ to the top half. Then the electrostatic potential, in polar co-ordinates, solves

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} = 0$$

subject to

$$\Phi(1,\theta) = \begin{cases} 0 & \text{if } -\pi < \theta \le 0 \\ 1 & \text{if } 0 \le \theta \le \pi \end{cases}$$

If we write the solution in complex exponential form, then

$$\begin{aligned} c_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\theta} f(\theta) d\theta \\ &= \frac{1}{2\pi}\int_0^{\pi} d\theta = \frac{1}{2} \end{aligned}$$

while for $n \neq 0$,

$$\begin{aligned} c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\theta} f(\theta) d\theta \\ &= \frac{1}{2\pi}\int_0^{\pi} e^{-in\theta} d\theta \\ &= \frac{e^{-in\pi} - 1}{-2\pi i n} \\ &= \frac{(-1)^n - 1}{-2\pi i n} \\ &= \begin{cases} \frac{1}{\pi i n} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases} \end{aligned}$$

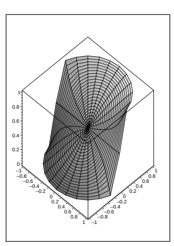
Thus

$$\Phi(r,\theta) = \frac{1}{2} + \sum_{\substack{n=-\infty \\ n \text{odd}}}^{\infty} \frac{1}{\pi i n} r^{|n|} e^{in\theta}$$

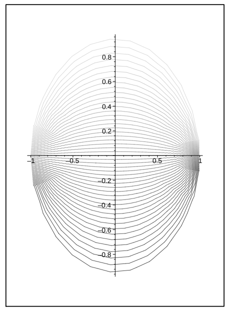
This can also be written in real form

$$\Phi(r,\theta) = \frac{1}{2} + \sum_{\substack{n=1 \\ n \text{odd}}}^{\infty} \frac{2}{\pi n} r^n e \sin(n\theta) = \frac{1}{2} + \sum_{n=0}^{\infty} \frac{2}{\pi(2n+1)} r^{(2n+1)} \sin((2n+1)\theta)$$

Here is picture of the solution



Lets plot the equipotential lines, i.e, the curves where Φ is constant.



Problem 4.2: Calculate the electrostatic potential inside an infinite cylinder of radius 1 which consists of four quarter round pieces, with one pair of opposite pieces grounded at 0 volts and the other pair kept at 10 volts.

The Poisson Kernel

We can think of the procedure of solving Laplaces equation with a given boundary condition as a kind of transformation that takes as input a function f on the boundary $\partial\Omega$ and produces as output the solution u to

$$\Delta u = 0 \quad \text{in } \Omega$$

satisfying

$$u = f \quad \text{on the boundary } \partial\Omega$$

This transformation is linear. In other words, if we take two boundary conditions f_1 and f_2 and find the corresponding solutions u_1 and u_2, then the solution with corresponding to the linear combination of boundary conditions $f = a_1 f_1 + a_2 f_2$ is the linear combination of solutions $u = a_1 u_1 + a_2 u_2$. We have already used this fact when we solved Laplace's equation on the rectangle. We wrote the boundary condition f as a sum of four pieces $f = f_1 + f_2 + f_3 + f_4$ and found the solution u_i of Laplace's equation corresponding to each f_i for $i = 1, \ldots, 4$. Then the solution corresponding to f was the sum $u_1 + u_2 + u_3 + u_4$.

In linear algebra, you learned that every linear transformation between finite dimensional vector spaces could be represented by a matrix. Something similar is true for Laplace's equation.

For definiteness, lets consider Laplace's equation on a disk of radius 1. The solution $u(r, \theta)$ in polar co-ordinates to Laplace's equation

$$\frac{\partial^2 u(r,\theta)}{\partial r^2} + \frac{1}{r}\frac{\partial u(r,\theta)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u(r,\theta)}{\partial \theta^2} = 0$$

satisfying the boundary condition

$$u(1,\theta) = f(\theta)$$

is

$$u(r,\theta) = \sum_{n=-\infty}^{\infty} c_n r^{|n|} e^{in\theta}$$

where

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\phi} f(\phi)\,d\phi.$$

Take the expression for c_n and insert it into the formula for u, then exchange the order of summation and integration. This gives

$$\begin{aligned} u(r,\theta) &= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\int_0^{2\pi} e^{-in\phi}f(\phi)d\phi r^{|n|}e^{in\theta} \\ &= \frac{1}{2\pi}\int_0^{2\pi}\sum_{n=-\infty}^{\infty} e^{in(\theta-\phi)}r^{|n|}f(\phi)d\phi \end{aligned}$$

But we can evaluate the sum using the formula for summing a geometric series. Recall that

$$\sum_{n=0}^{\infty} w^n = \frac{1}{1-w}$$

if $|w|<1$. Below we will break the sum $\sum_{-\infty}^{\infty}$ into two sums and in doing so count the $n=0$ term twice. This explains the -1 in the formula below.

$$\begin{aligned} \sum_{n=-\infty}^{\infty} e^{in(\theta-\phi)}r^{|n|} &= \sum_{n=0}^{\infty} e^{in(\theta-\phi)}r^n + \sum_{n=-\infty}^{0} e^{in(\theta-\phi)}r^{-n} - 1 \\ &= \sum_{n=0}^{\infty} e^{in(\theta-\phi)}r^n + \sum_{n=0}^{\infty} e^{-in(\theta-\phi)}r^n - 1 \\ &= \frac{1}{1-e^{i(\theta-\phi)}r} + \frac{1}{1-e^{-i(\theta-\phi)}r} - 1 \\ &= \frac{1-e^{-i(\theta-\phi)}r + 1 - e^{i(\theta-\phi)}r}{2-e^{i(\theta-\phi)}r - e^{-i(\theta-\phi)}r} - 1 \\ &= \frac{2-2r\cos((\theta-\phi)}{1-2r\cos((\theta-\phi)+r^2} - 1 \\ &= \frac{2-2r\cos((\theta-\phi)-1+2r\cos((\theta-\phi)-r^2}{1-2r\cos((\theta-\phi)+r^2} \\ &= \frac{1-r^2}{1-2r\cos((\theta-\phi)+r^2} \end{aligned}$$

Define the function $P(r,\theta,\phi)$ (the Poisson kernel) to be

$$P(r,\theta,\phi) = \frac{1}{2\pi}\frac{1-r^2}{1-2r\cos((\theta-\phi)+r^2}$$

Then we get this beautiful formula, expressing the solution $u(r,\theta)$ directly in terms of the boundary condition f:

$$u(r,\theta) = \int_0^{2\pi} P(r,\theta,\phi)f(\phi)d\phi$$

Thus P is somewhat analogous to a matrix.

The inhomogeneous Laplace equation $-\Delta u = f$ with homogeneous boundary conditions

We will now consider Laplace's equation with an inhomogeneous term, but with homogeneous (we will consider only Dirichlet) boundary conditions. This equation arises, for example, in electrostatics when there is a non-zero charge density.

To start off, lets consider the case of one space dimension. For simplicity, lets assume that we are working on the interval $[0, 1]$ Then we want to solve

$$-\Delta u(x) = -u''(x) = f(x) \quad \text{for} 0 < x < 1$$

subject to the Dirichlet boundary conditions

$$u(0) = u(1) = 0$$

To solve this equation, we expand every function in a series of eigenfunctions of $-\Delta = -d^2/dx^2$ with Dirichlet boundary condition. In other words, we use a Fourier sine series. First we expand the given function f

$$f(x) = \sum_{n=1}^{\infty} f_n \sin(n\pi x)$$

with

$$f_n = 2\int_0^1 \sin(n\pi x) f(x) dx$$

The unknown function u has an expansion

$$u(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$$

where the coefficients b_n are undetermined. Since we have chosen a sine series, u will satisfy the Dirichlet boundary conditions. The equation now reads

$$-u''(x) = \sum_{n=1}^{\infty} -b_n \sin''(n\pi x) = \sum_{n=1}^{\infty} n^2\pi^2 b_n \sin(n\pi x) = \sum_{n=1}^{\infty} f_n \sin(n\pi x)$$

So we see that u is a solution if

$$b_n = \frac{1}{n^2\pi^2} f_n$$

Thus

$$u(x) = \sum_{n=1}^{\infty} \frac{1}{n^2\pi^2} f_n \sin(n\pi x)$$

But wait, isn't there an easier way to solve the equation $-u'' = f$? We could simply integrate both sides of the equation twice. This gives

$$\begin{aligned} u(x) &= \alpha x + \beta - \int_0^x \int_0^s f(r) dr ds \\ &= \alpha x + \beta - \int_0^x (r - x) f(r) dr \end{aligned}$$

(Here we changed the order of integration, and then performed the inside intregral.) The arbitrary constants α and β can be chosen to satisfy the Dirichlet boundary conditions. Thus

$$u(0) = \beta = 0$$

and

$$u(1) = \alpha + 0 - \int_0^1 (r - 1) f(r) dr = 0$$

so

$$\alpha = -\int_0^1 (r - 1) f(r) dr$$

The solution can therefore be written

$$\begin{aligned}u(x) &= -x\int_0^1 (r-1)f(r)dr + \int_0^x (r-x)f(r)dr\\ &= \int_0^x r(1-x)f(r)dr + \int_x^1 x(r-1)f(r)dr\\ &= \int_0^1 G(x,r)f(r)dr\end{aligned}$$

where

$$G(x,r) = \begin{cases} r(1-x) & \text{if } r < x\\ x(1-r) & \text{if } r > x\end{cases}$$

The function $G(x,r)$ is called the Green's function for this equation.

We therefore have two different methods of solving the same equation—the eigenfunction expansion (Fourier series) method and the Green's function method. Of course they must produce the same answer. In fact, if we start with the series and substitute in the formulas for the f_n we get

$$\begin{aligned}u(x) &= \sum_{n=1}^{\infty} \frac{1}{n^2\pi^2} f_n \sin(n\pi x)\\ &= \sum_{n=1}^{\infty} \frac{1}{n^2\pi^2} 2\int_0^1 \sin(n\pi r)f(r)dr \sin(n\pi x)\\ &= \int_0^1 \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}\sin(n\pi x)\sin(n\pi r)f(r)dr\end{aligned}$$

and one can verify that

$$G(x,r) = \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}\sin(n\pi x)\sin(n\pi r) \tag{4.2}$$

Example

Consider the equation

$$-u''(x) = e^x$$

for $0 < x < 1$ subject to $u(0) = u(1) = 0$. If we want to solve this using eigenfunctions, we must first expand $f(x) = e^x$ in a sine series. We obtain

$$f_n = 2\int_0^1 \sin(n\pi x)e^x dx = \frac{n\pi(1-e(-1)^n)}{1+n^2\pi^2}$$

so that

$$u(x) = \sum_{n=1}^{\infty} \frac{n\pi(1-e(-1)^n)}{(1+n^2\pi^2)(n^2\pi^2)}\sin(n\pi x)$$

On the other hand, if we want to use the Green's function we may write

$$u(x) = \int_0^x r(1-x)e^r dr + \int_x^r x(r-1)e^r dr = e^x(-2x^2+4x-1)+1-x-ex$$

Problem 4.3: Verify the identity (4.2) by expanding $G(x,r)$ for fixed x in a sine series in the r variable.

Problem 4.4: Solve the equation

$$-u''(x) = x$$

with Dirichlet boundary conditions at $x = 0$ and $x = 1$ using both methods.

Now we consider two space dimensions. Now we are given some domain Ω and we wish to solve

$$-\Delta u(x, y) = f(x, y)$$

for $(x, y) \in \Omega$ with boundary conditions

$$u(x, y) = 0$$

for $(x, y) \in \partial\Omega$.

In principle, this can be solved with an eigenfunction expansion, just as in before. If we know the eigenfunctions $\phi_{n,m}(x, y)$ and eigenvalues $\lambda_{n,m}$ of $-\Delta$ with Dirichlet boundary conditions, we can expand

$$f(x, y) = \sum_n \sum_m f_{n,m}\phi_{n,m}(x, y)$$

where

$$f_{n,m} = \langle \phi_{n,m}, f\rangle / \langle \phi_{n,m}, \phi_{n,m}\rangle = \int_\Omega \phi_{n,m}(x, y) f(x, y) dxdy \Big/ \int_\Omega \phi_{n,m}^2(x, y) dxdy$$

Then if we write

$$u(x, y) = \sum_n \sum_m b_{n,m}\phi_{n,m}(x, y)$$

for unknown coefficients $b_{n,m}$, the boundary conditions will hold, and the equation reads

$$-\Delta u = \sum_n \sum_m b_{n,m}(-\Delta\phi_{n,m})(x, y) = \sum_n \sum_m b_{n,m}\lambda_{n,m}\phi_{n,m}(x, y) = \sum_n \sum_m f_{n,m}\phi_{n,m}(x, y)$$

Thus the unknown coefficients are given by

$$b_{n,m} = \frac{1}{\lambda_{n,m}} f_{n,m}$$

and so

$$u(x, y) = \sum_n \sum_m \frac{1}{\lambda_{n,m}} f_{n,m}\phi_{n,m}(x, y)$$

Of course, to actually use this method in practice, we must know the eigenfunctions and eigenvalues. At this point we know them explicitly only for rectangles in two dimensions (and not for disks). So even though we can solve the homogeneous Laplace's equation on a disk, we can't yet solve the inhomogeneous equation.

Lets solve

$$-\Delta u(x, y) = 1$$

for $0 < x < 1$ and $0 < y < 1$ subject to

$$u(x, 0) = u(x, 1) = u(0, y) = u(1, y) = 0.$$

The eigenfunctions are $\phi_{n,m}(x, y) = \sin(n\pi x)\sin(n\pi y)$ and the eigenvalues are $\lambda_{n,m} = \pi^2(n^2 + m^2)$. We first expand

$$1 = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} 4\frac{((-1)^n - 1)((-1)^m - 1)}{\pi^2 nm} \sin(n\pi x)\sin(n\pi y)$$

Thus we obtain

$$u(x,y)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}4\frac{((-1)^n-1)((-1)^m-1)}{\pi^4 nm(n^2+m^2)}\sin(n\pi x)\sin(n\pi y)$$

What about the Green's function method in two dimensions? There is a Green's function that we may compute in terms of the eigenfunctions as

$$G(x,y,s,t)=\sum_{n,m}\frac{\phi_{n,m}(x,y)\phi_{n,m}(s,t)}{\lambda_{n,m}\langle\phi_{n,m},\phi_{n,m}\rangle}$$

This function can be used to write the solution u as

$$u(x,y)=\int_{\Omega}G(x,y,s,t)f(s,t)dsdt$$

There are also ways (in addition to the eigenfunction formula above) to compute the Green's function in various situations. Unfortunately we won't have time in this course to explore this.

Problem 4.5: Find the solution $u(x,y)$ to

$$-\Delta u(x,y)=xy$$

for $0<x<1$ and $0<y<1$ that satisfies Dirichlet boundary conditions.

Problem 4.6: Find the solution to

$$-\Delta u(x,y)=1$$

that satisfies the non-homogeneous boundary conditions

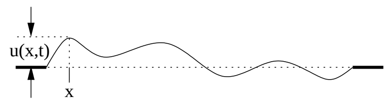
$$\begin{aligned}u(x,0)&=x \quad \text{for } 0\le x\le 1\\ u(1,y)&=1 \quad \text{for } 0\le y\le 1\\ u(x,1)&=1 \quad \text{for } 0\le x\le 1\\ u(0,y)&=1 \quad \text{for } 0\le y\le 1\end{aligned}$$

Hint: let $u_1(x,y)$ be the solution of $-\Delta u=1$ with Dirichlet boundary conditions. Then consider $v(x,y)=u(x,y)-u_1(x,y)$. The function v will satisfy Laplace's equation with the same boundary conditions, but without the nonhomogeneous term.

The wave equation

The wave equation is governs the behaviour of vibrating strings, columns of air, water waves, electromagnetic fields and many other wave phenomena in nature.

Consider a vibrating string of length l, fastened down at either end.



At a given moment of time, let $u(x,t)$ be the height of the string above its relaxed position. If the amplitude of the vibrations is not too large, then to a good approximation, $u(x,t)$ function will satisfy the wave equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}$$

and the Dirichlet boundary conditions

$$u(0,t) = u(l,t) = 0,$$

reflecting the fact that the string is tied down at the ends. Since the wave equation is second order in time, we need two initial conditions

$$u(x,0) = u_0(x)$$
$$\frac{\partial u}{\partial t}(x,0) = v_0(x)$$

In other words we must specify the initial position and the initial velocity of the string. The wave equation, together with boundary conditions and initial conditions determine the function $u(x,t)$ completely.

This equation must be regarded as an approximation. It does not take into account dissipative effects and is not valid for large amplitude vibrations. Nevertheless, it captures much of the behaviour of a real string.

D'Alembert's solution on the whole line: travelling waves

We will begin the study of the nature of the solutions to the wave equation by considering the idealized situation where the string is infinitely long. In this case there are no boundary conditions, just the wave equation and the initial conditions.

Let $\phi(x)$ be an arbitrary (twice differentiable) function. Then I claim that $u(x,t) = \phi(x-ct)$ is a solution to the wave equation. To see this, we simply differentiate. By the chain rule

$$\frac{\partial u(x,t)}{\partial t} = \phi'(x-ct)(-c)$$

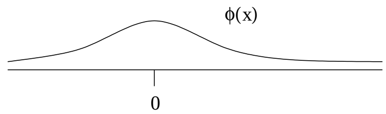
$$\frac{\partial^2 u(x,t)}{\partial t^2} = \phi''(x-ct)(-c)^2 = c^2\phi''(x-ct)$$

while

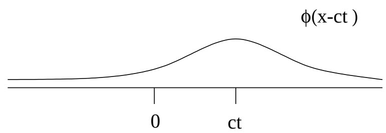
$$\frac{\partial^2 u(x,t)}{\partial x^2} = \phi''(x-ct)$$

so that $\partial^2 u(x,t)/\partial t^2 = c^2 \partial^2 u(x,t)/\partial x^2$ as claimed.

The function $\phi(x-ct)$ represents a wave travelling to the right with speed c. To see this, suppose that $\phi(x)$ is a bump centered at the origin



Then $\phi(x-ct)$ is the same bump shifted to the right by and amount ct



The speed of the wave is $ct/t = c$.

A similar calculation shows that $u(x,t) = \phi(x+ct)$, that is, a wave travelling to the left, also is a solution of the wave equation. By linearity, a sum of left moving and right moving waves

$$u(x,t) = \phi_1(x-ct) + \phi_2(x+ct)$$

is also a solution of the wave equation, for any choice of ϕ_1 and ϕ_2

It turns out that for an infinite string, every solution of the wave equation is of this form. To see this lets introduce the new co-ordinates

$$\begin{aligned} r &= x+ct \qquad & x &= (r+s)/2 \\ s &= x-ct \qquad & t &= (r-s)/(2c) \end{aligned}$$

Then, by the chain rule in two dimensions

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial r} = \frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{2c}\frac{\partial u}{\partial t}$$

or, briefly,

$$2c\frac{\partial}{\partial r} = \frac{\partial}{\partial t} + c\frac{\partial}{\partial x}$$

Similarly,

$$2c\frac{\partial}{\partial s} = \frac{\partial}{\partial t} - c\frac{\partial}{\partial x}$$

Therefore

$$4c^2\frac{\partial^2}{\partial r \partial s} = \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) = \frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}$$

Now the combination of derivatives on the right side is exactly what is zero if the wave equation holds. So every solution of the wave equation satisfies

$$\frac{\partial^2}{\partial r \partial s}u = \frac{\partial}{\partial r}\frac{\partial u}{\partial s} = 0$$

This says that the r derivative of $\partial u/\partial s$ is zero, so $\partial u/\partial s$ must depend only on s. Thus

$$\frac{\partial u}{\partial s} = \psi_1(s)$$

which implies

$$u = \int \psi_1(s)ds + \psi_2(r)$$

This shows that u is a sum of a function r and a function of s, that is, a sum of a right moving and a left moving wave.

Lets show that every set of initial conditions can be satisfied by such a function. To start, lets consider initial conditions of the form

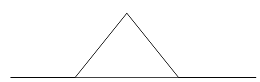
$$u(x,0) = u_0(x)$$
$$\frac{\partial u(x,0)}{\partial t} = 0$$

This represents a stationary initial condition with shape $u_0(x)$. We need to find $\phi_1(x)$ and $\phi_2(x)$ such that

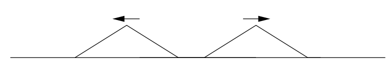
$$\phi_1(x) + \phi_2(x) = u_0(x)$$
$$-c\phi_1'(x) + c\phi_2'(x) = 0$$

This can be satisfied with $\phi_1(x) = \phi_2(x) = u_0(x)/2$. In other words the initial shape $u_0(x)$ will split into two identical shapes of half the height moving to the left and to the right, and

$$u(x,t) = \frac{u_0(x-ct)}{2} + \frac{u_0(x+ct)}{2}$$



initial shape



Next, suppose we want to satisfy the initial conditions

$$u(x,0) = 0$$
$$\frac{\partial u(x,0)}{\partial t} = v_0(x)$$

This represents an initial condition that is flat, but has an initial velocity given by v_0. We now need to find $\phi_1(x)$ and $\phi_2(x)$ such that

$$\phi_1(x) + \phi_2(x) = 0$$
$$-c\phi_1'(x) + c\phi_2'(x) = v_0(x)$$

The first equation says that $\phi_2 = -\phi_1$ and thus the second equation can be rewritten

$$\phi_1'(x) = \frac{-1}{2c} v_0(x)$$

or

$$\phi_1(x) = \frac{-1}{2c}\int_0^x v_0(r)dr + C$$

Thus

$$\begin{aligned}\phi_1(x-ct) + \phi_2(x+ct) &= \frac{-1}{2c}\int_0^{x-ct} v_0(r)dr + C + \frac{1}{2c}\int_0^{x+ct} v_0(r)dr - C \\ &= \frac{1}{2c}\int_{x-ct}^{x+ct} v_0(r)dr\end{aligned}$$

Finally, an arbitrary initial condition

$$u(x,0) = u_0(x)$$
$$\frac{\partial u(x,0)}{\partial t} = v_0(x)$$

is satisfied by the solution obtained by adding together the two special cases.

$$u(x,t) = \frac{u_0(x-ct)}{2} + \frac{u_0(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} v_0(r)dr$$

An interesting point to notice is that the formula above makes sense, even if the functions u_0 and v_0 are not differentiable. In this case, $u(x,t)$ is still a sum of a right moving and a left moving wave, but the shape of the wave may have corners and even jumps. It is still natural to call such a function a (weak) solution of the wave equation. In fact, the illustration above with the triangular wave is a weak solution. Technically, it doesn't have two derivatives with respect to x or t, so strictly speaking it does not satisfy the wave equation.

Problem 5.1: Suppose an infinite string is hit with a hammer, so that the initial conditions are given by $u_0(x) = 0$ and $v_0(x) = \begin{cases} 1 & \text{if } -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$. Find the shape of the string for all later times.

Sine series expansion

We return now to the finite string which is held down at both ends. Thus we wish to solve

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2\frac{\partial^2 u(x,t)}{\partial x^2}$$

subject to the boundary conditions

$$u(0,t) = u(l,t) = 0,$$

and the initial conditions

$$u(x,0) = u_0(x)$$
$$\frac{\partial u}{\partial t}(x,0) = v_0(x)$$

Given the boundary conditions, it is natural to try to find the solution in the form of a sine series. Let

$$u(x,t) = \sum_{n=1}^{\infty}\beta_n(t)sin(n\pi x/l)$$

This will satisfy the correct boundary conditions. It will satisfy the equation if

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \sum_{n=1}^{\infty}\beta_n''(t)sin(n\pi x/l)$$

is equal to

$$c^2\frac{\partial^2 u(x,t)}{\partial x^2}\sum_{n=1}^{\infty} -(cn\pi/l)^2\beta_n(t)sin(n\pi x/l)$$

In other words, we need

$$\beta_n''(t) = -(cn\pi/l)^2\beta_n(t)$$

This has solution

$$\beta_n(t) = a_n\cos(cn\pi t/l) + b_n\sin(cn\pi t/l)$$

for arbitrary constants a_n and b_n. The constants are determined by the initial conditions:

$$u(x,0) = \sum_{n=1}^{\infty}\beta_n(0)\sin(n\pi x/l) = \sum_{n=1}^{\infty}a_n\sin(n\pi x/l) = u_0(x)$$

so that

$$a_n = \frac{2}{l}\int_0^l \sin(n\pi x/l)u_0(x)dx$$

and

$$\frac{\partial u(x,0)}{\partial t} = \sum_{n=1}^{\infty}\beta_n'(0)\sin(n\pi x/l) = \sum_{n=1}^{\infty}cn\pi b_n/l\sin(n\pi x/l) = v_0(x)$$

so that

$$b_n = \frac{2}{cn\pi}\int_0^l \sin(n\pi x/l)v_0(x)dx$$

The final form of the solution is therefore

$$u(x,t) = \sum_{n=1}^{\infty}\Big(a_n\cos(cn\pi t/l) + b_n\sin(cn\pi t/l)\Big)\sin(n\pi x/l)$$

where a_n and b_n are given by the integrals above.

Problem 5.2: Find the solution of the wave equation on the interval $[0,1]$ with Dirichlet boundary conditions and initial conditions

$$u_0 = \begin{cases} x & \text{if } 0 \le x \le 1/2 \\ 1-x & \text{if } 1/2 \le x \le 1 \end{cases}$$

and $v_0 = 0$.

Problem 5.3: Find the solution of the wave equation on the interval $[0,1]$ with Dirichlet boundary conditions and initial conditions $u_0 = 0$ and

$$v_0 = \begin{cases} x & \text{if } 0 \le x \le 1/2 \\ 1-x & \text{if } 1/2 \le x \le 1 \end{cases}.$$

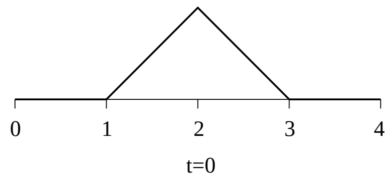
Travelling waves for a finite string

Suppose we consider a finite string, with an initial disturbance that is restricted to the middle part of the string. For definiteness, suppose that the string lies between 0 and 4 and the initial conditions are

$$u(x,0) = u_0(x) = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ x-1 & \text{if } 1 \le x < 2 \\ 3-x & \text{if } 2 \le x < 3 \\ 0 & \text{if } 3 \le x \le 4 \end{cases}$$

$$\frac{\partial u(x,0)}{\partial t} = v_0(x) = 0$$

Here is a picture of the initial position.



To compute the solution for later times, we use the sine series solution. Let's assume that the speed $c = 1$. Then

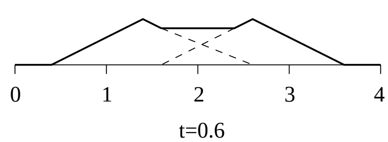
$$u(x,t) = \sum_{n=1}^{\infty} \Big(a_n \cos(cn\pi t/l) + b_n \sin(cn\pi t/l) \Big) \sin(n\pi x/l)$$

with $c = 1$, $l = 4$, $b_n = 0$ for every n (since $v_0 = 0$) and

$$\begin{aligned} a_n &= \frac{1}{2}\int_0^4 \sin(n\pi x/4) u_0(x)dx \\ &= \frac{1}{2}\int_1^2 (x-1)\sin(n\pi x/4)dx + \frac{1}{2}\int_2^3 (3-x)\sin(n\pi x/4)dx \\ &= \frac{8(2\sin(n\pi/2) - \sin(n\pi/4) - \sin(3n\pi/4))}{n^2\pi^2} \end{aligned}$$

Although the sine series solution is mathematically valid, its pretty hard to see from the formula what happens as time goes on.

Lets first think about what happens for short times, before the disturbance has had a chance to reach the end of the string. For these times, the d'Alembert solutions consisting of two waves of half the height moving to the left and the right will satisfy the boundary condition for eh finite string. So this must be the solution for the finite string too.

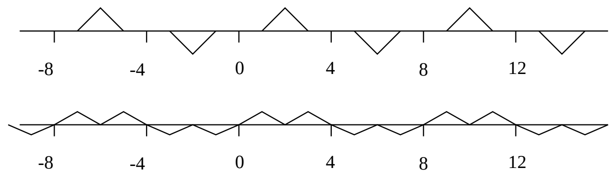


However, at $t = 1$ the edge of the disturbance reaches the endpoints, and so the d'Alembert solution no longer satisfies the boundary conditions. What happens?

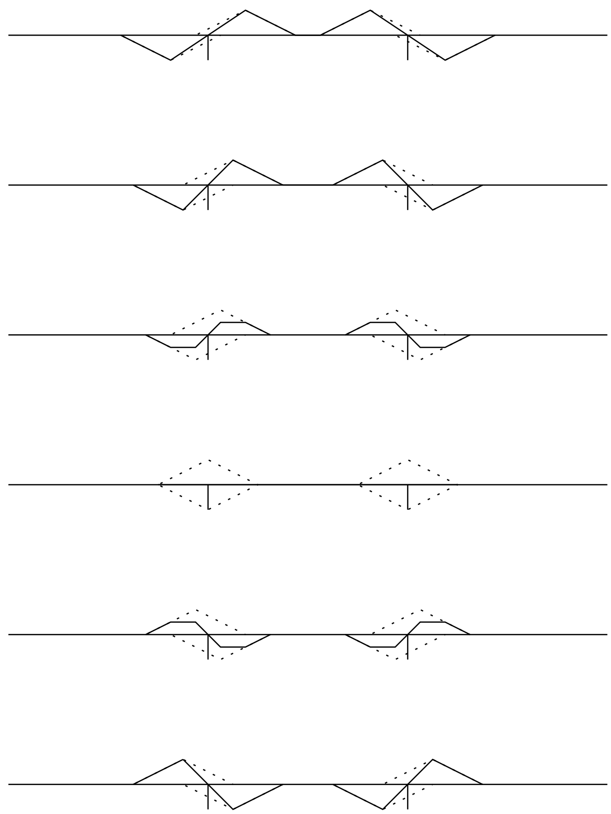
The key to understanding this is to realize that the sine series solution, although it only has physical relevance for x in the interval $0 \le x \le 4$, actually makes mathematical sense for all x. Moreover, as a function of x on the whole real line, it satisfies the wave equation. Therefore it must be a sum of a right moving and a left moving wave. The initial condition

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin(n\pi x/4)$$

when considered as a function on the whole real line, is the function on $[0,4]$, extended to be an odd function on $[-4,4]$ and then extended periodically with period 8. Here is a picture.



In the lower picture above, the wave has split into two and has moved to the right and left. Let's examine what happens next near the boundary.



Here we see the inverted wave from the right and the left moving into the physical region. What we see in the physical region is a reflection, with an inverted reflected wave.

We can also use trig formulas to see that $u(x,t)$ is the sum of a right moving and a left moving wave. In fact $\cos(n\pi t/4)\sin(n\pi x/4) = (\sin(n\pi(x+t)/4) + \sin(n\pi(x-t)/4))/2$ so that

$$u(x,t) = \sum_{n=1}^{\infty} a_n \cos(n\pi t/4)\sin(n\pi x/4) = \sum_{n=1}^{\infty} \frac{a_n}{2}\sin(n\pi(x+t)/4) + \sum_{n=1}^{\infty} \frac{a_n}{2}\sin(n\pi(x-t)/4)$$

Standing waves

The solutions of the wave equation of the form

$$\big(a_n\cos(cn\pi t/l) + b_n\sin(cn\pi t/l)\big)\sin(n\pi x/l)$$

are called standing waves. These are solutions whose sine series has only one non-zero term. Saying that an arbitrary solution can be expanded in a sine series is the same as saying that any solution is a superposition of standing waves.

Notice that the shape of a standing wave is just a fixed shape, namely that of the eigenfunction $\sin(n\pi x/l)$ times a factor oscillating sinusoidally in time with frequency $cn\pi/l$. For a given string, only a discrete set of frequencies are possible, namely $cn\pi/l$ for $n = 1, 2, 3, \ldots$. The frequency corresponding to $n = 1$ is called the fundamental tone and the rest are called overtones. When you hear the sound of a single string on an instrument like a piano, you are actually hearing a combination of the fundamental tone and overtones.

It is interesting to compare the possible frequencies that can be emitted from a single string on a piano (say an A with frequency 110) with the fundamental frequencies of the other strings. These are tuned according to the principle of equal tempering. Mathematically this means that the k'th note above A has frequency $2^{k/12} \cdot 110$ (although I doubt J.S. Bach would have expressed it this way.)

Notice that each overtone matches pretty well (though not exactly) with one of the tempered notes . . . until we reach the seventh. For this reason, the hammer on a piano hits the string one seventh of the way down the string. This is because the standing wave corresponding to this overtone would have a zero (or node) at this point. By hitting the string exactly at this point, we are minimizing the amplitude of this term in the sine series, so we won't hear much of this offending overtone!

Note	Tempered frequency($2^{k/12} \cdot 110$)	Overtone frequency($n \cdot 110$)
A	110.0000000	110
$B\flat$	116.5409403	
B	123.4708253	
C	130.8127827	
$C\sharp$	138.5913155	
D	146.8323839	
$E\flat$	155.5634918	
E	164.8137785	
F	174.6141157	
$F\sharp$	184.9972114	
G	195.9977180	
$A\flat$	207.6523488	
A	220.0000000	220
$B\flat$	233.0818807	
B	246.9416506	
C	261.6255653	
$C\sharp$	277.1826310	
D	293.6647679	
$E\flat$	311.1269836	
E	329.6275569	330
F	349.2282314	
$F\sharp$	369.9944228	
G	391.9954359	
$A\flat$	415.3046975	
A	440.0000000	440
$B\flat$	466.1637614	
B	493.8833011	
C	523.2511306	
$C\sharp$	554.3652620	550
D	587.3295358	
$E\flat$	622.2539673	

Note	Tempered frequency($2^{k/12} \cdot 110$)	Overtone frequency($n \cdot 110$)
E	659.2551139	660
F	698.4564629	
$F\sharp$	739.9888456	
G	783.9908718	770
$A\flat$	830.6093950	
A	880.0000000	880

The wave equation in two space dimensions

The wave equation in two dimensions describes the vibrations of a drum, or water waves on a pond. Given a region Ω in the plane, we wish to solve

$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

with boundary conditions

$$u(x, y, t) = 0 \quad \text{for } (x, y) \in \partial\Omega$$

and initial conditions

$$u(x, y, 0) = u_0(x, y)$$
$$\frac{\partial u}{\partial t}(x, y, 0) = v_0(x, y)$$

We can solve this equation, if we know the Dirichlet eigenfunctions of $-\Delta$. Suppose they are given by $\phi_{n,m}$, with eigenvalues $\lambda_{n,m}$. In other words,

$$-\Delta\phi_{n,m} = \lambda_{n,m}\phi_{n,m}$$

and

$$\phi_{n,m}(x, y, t) = 0 \quad \text{for } (x, y) \in \partial\Omega$$

Then we can write the solution u as a series

$$u(x, y, t) = \sum_{n,m} \beta_{n,m}(t)\phi_{n,m}(x, y)$$

Since each $\phi_{n,m}(x, y)$ vanishes on the boundary, so will u, as required. We compute that

$$\frac{\partial^2 u}{\partial t^2} = \sum_{n,m} \beta''_{n,m}(t)\phi_{n,m}(x, y)$$

and

$$c^2\Delta u = \sum_{n,m} \beta_{n,m}(t)c^2\Delta\phi_{n,m}(x, y) = \sum_{n,m} -c^2\lambda_{n,m}\beta_{n,m}(t)\phi_{n,m}(x, y)$$

Therefore, the wave equation holds if

$$\beta''_{n,m}(t) = -c^2\lambda_{n,m}\beta_{n,m}(t)$$

which, in turn, holds if

$$\beta_{n,m}(t) = a_{n,m}\cos(c\sqrt{\lambda_{n,m}}t) + b_{n,m}\sin(c\sqrt{\lambda_{n,m}}t)$$

for arbitrary constants $a_{n,m}$ and $b_{n,m}$. These constants can be determined using the initial conditions.

$$u(x, y, 0) = \sum_{n,m} \beta_{n,m}(0)\phi_{n,m}(x, y) = \sum_{n,m} a_{n,m}\phi_{n,m}(x, y) = u_0(x, y)$$

so that

$$a_{n,m} = \langle \phi_{n,m}, u_0 \rangle / \langle \phi_{n,m}, \phi_{n,m} \rangle$$
$$= \int\int_\Omega \phi_{n,m}(x,y) u_0(x,y) dxdy \Big/ \int\int_\Omega \phi_{n,m}^2(x,y) dxdy$$

and

$$\frac{\partial u}{\partial t}(x,y,0) = \sum_{n,m} \beta'_{n,m}(0)\phi_{n,m}(x,y) = \sum_{n,m} c\sqrt{\lambda_{n,m}} b_{n,m}\phi_{n,m}(x,y) = v_0(x,y)$$

so that

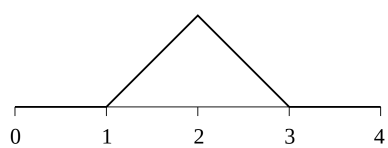
$$b_{n,m} = (c\sqrt{\lambda_{n,m}})^{-1} \langle \phi_{n,m}, v_0 \rangle / \langle \phi_{n,m}, \phi_{n,m} \rangle$$
$$= (c\sqrt{\lambda_{n,m}})^{-1} \int\int_\Omega \phi_{n,m}(x,y) u_0(x,y) dxdy \Big/ \int\int_\Omega \phi_{n,m}^2(x,y) dxdy$$

An example

Lets solve the wave equation on the square $0 \le x \le 4$ and $0 \le y \le 4$ with Dirichlet boundary conditions and initial condition given by the product

$$u(x,y,0) = u_0(x)u_0(y)$$

where $u_0(x)$ is the triangular wave in the center of the interval that we considered above.



and

$$\frac{\partial u}{\partial t}(x,y,0) = 0$$

The eigenfunctions for this problem are the double sine functions

$$\phi_{n,m}(x,y) = \sin(n\pi x/4)\sin(m\pi y/4)$$

and the eigenvalues are

$$\lambda_{n,m} = \pi^2(n^2+m^2)/4^2$$

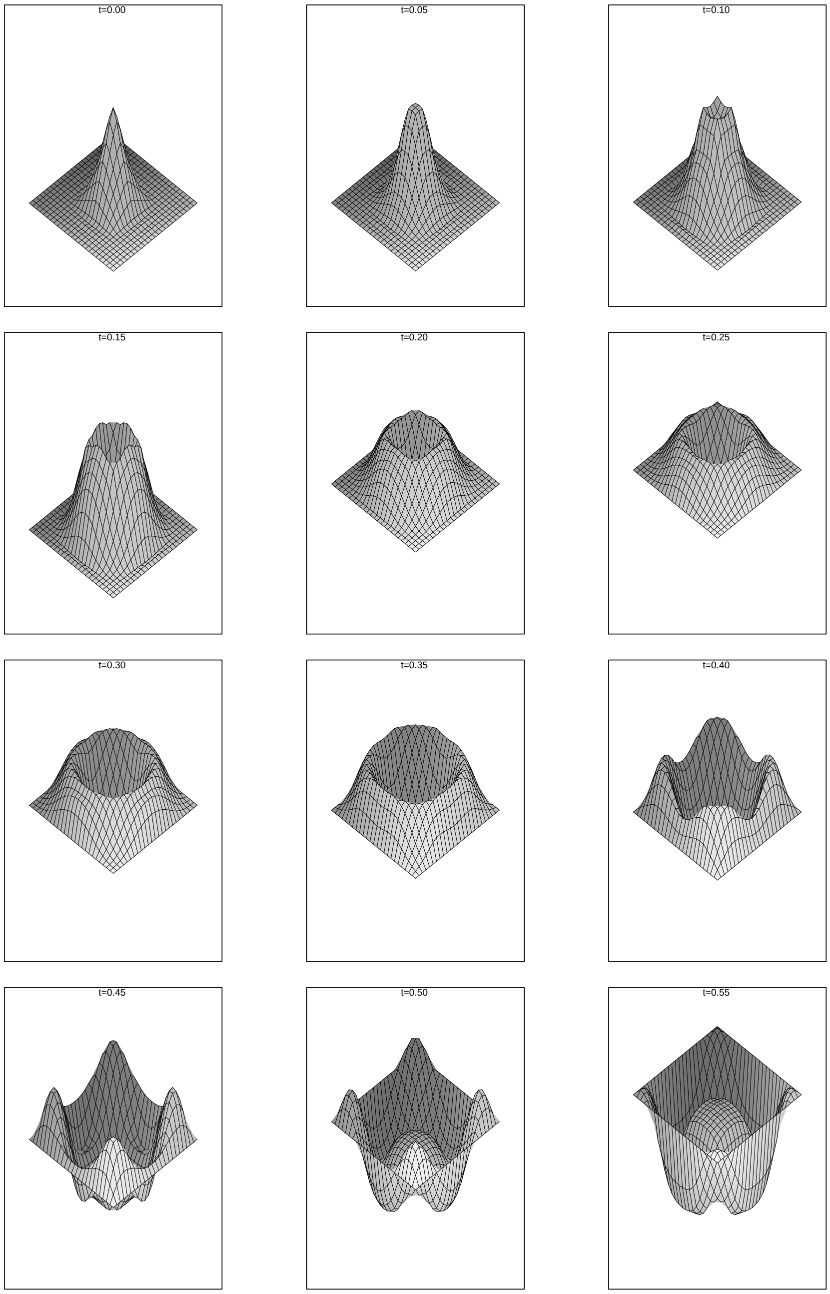
Since the initial velocity is zero, all the $b_{n,m}$'s are zero, and

$$a_{n,m} = \left(\frac{2}{4}\right)^2 \int_0^4\int_0^4 \sin(n\pi x/4)\sin(m\pi y/4)u_0(x)u_0(y)dxdy$$
$$= \left(\frac{2}{4}\int_0^4 \sin(n\pi x/4)u_0(x)dx\right)\left(\frac{2}{4}\int_0^4 \sin(m\pi y/4)u_0(y)dy\right)$$
$$= \left(\frac{8(2\sin(n\pi/2) - \sin(n\pi/4) - \sin(3n\pi/4))}{n^2\pi^2}\right)\left(\frac{8(2\sin(m\pi/2) - \sin(m\pi/4) - \sin(3m\pi/4))}{m^2\pi^2}\right)$$

With this definition of $a_{n,m}$ we have

$$u(x,y,t) = \sum_{n=1}^\infty\sum_{m=1}^\infty a_{n,m}\cos(\pi\sqrt{n^2+m^2}t/4)\sin(n\pi x/4)\sin(m\pi y/4)$$

Here are plots of the vibrating membrane for some small values of t.



Problem 5.4: Solve the wave equation

$$\frac{\partial^2 u}{\partial t^2} = 4\Delta u$$

in the rectangle $0 \le x \le 2$, $0 \le y \le 1$ with Dirichlet boundary conditions and with initial conditions

$$u(x,y,0) = 0$$
$$\frac{\partial u}{\partial t}(x,y,0) = xy$$

Kirchhoff's formula and Huygens' principle

We now want to consider a wave propagating in all of three dimensional space. For example, think of a sound wave. In this case $u(x,y,z,t)$ is the pressure at position x,y,z and time t. The function u will satisfy the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

with some initial conditions

$$u(x,y,z,0) = u_0(x,y,z)$$

$$\frac{\partial u}{\partial t}(x,y,z,0) = v_0(x,y,z)$$

Is there a formula, analogous to d'Alembert's formula, that gives the solution directly in terms of the initial data? There is such a formula, called Kirchhoff's formula. It is

$$u(\mathbf{x},t) = \frac{1}{4\pi c^2 t} \iint_{\mathbf{y}:|\mathbf{x}-\mathbf{y}|=ct} u_0(\mathbf{y}) dS + \frac{\partial}{\partial t} \left[\frac{1}{4\pi c^2 t} \iint_{\mathbf{y}:|\mathbf{x}-\mathbf{y}|=ct} v_0(\mathbf{y}) dS \right]$$

I've changed notation slightly: in this formula $\mathbf{x}$ and $\mathbf{y}$ denote three dimensional vectors.

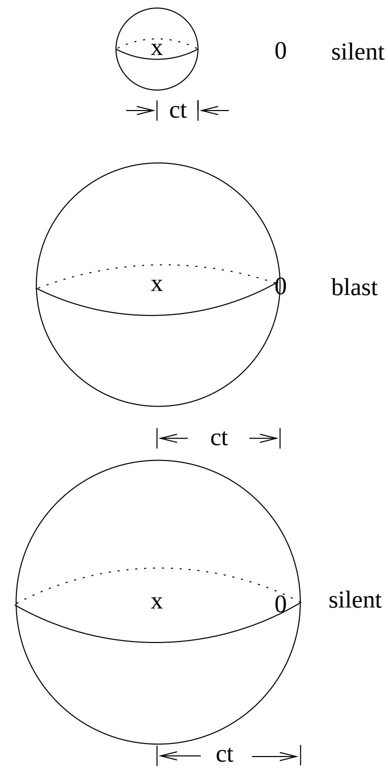
I won't go through the derivation of this formula. It can be found in *Partial Differential Equations, an Introduction* by Walter Strauss.

The most important feature of this formula is the fact that the integral is a *surface* integral over a sphere of radius ct centered at $\mathbf{x}$. Suppose that a tuba player is sitting at position $\mathbf{y} = (0,0,0)$ and lets out a blast at time $t = 0$. This means that the initial conditions $u_0(\mathbf{y})$ and $v_0(\mathbf{y})$ are concentrated close to $\mathbf{y} = (0,0,0)$.

What does a listener at position $\mathbf{x}$ hear? The surface integrals

$$\iint_{\mathbf{y}:|\mathbf{x}-\mathbf{y}|=ct} u_0(\mathbf{y}) dS, \qquad \iint_{\mathbf{y}:|\mathbf{x}-\mathbf{y}|=ct} v_0(\mathbf{y}) dS$$

is zero unless the sphere $\mathbf{y} : |\mathbf{x}-\mathbf{y}| = ct$ passes through (or close to) zero. For small t this will not be the case, since $\mathbf{y} : |\mathbf{x}-\mathbf{y}| = ct$ is a small sphere centered at $\mathbf{x}$. So the integrals in the formula for $u(\mathbf{x},t)$ will be zero. Then, when ct is exactly equal to the distance from $\mathbf{x}$ to zero, the integral will be non-zero, and so $u(\mathbf{x},t)$ will be non-zero. At this instant in time, the listener at $\mathbf{x}$ hears the blast. But then, for later times, the sphere $\mathbf{y} : |\mathbf{x}-\mathbf{y}| = ct$ is too big, and doesn't pass through zero anymore. Thus $u(\mathbf{x},t)$ is zero and the listener hears silence again.



Another way of saying this, is that at any time t, the sound wave for the blast is zero, except on the shell of points that are a distance exactly ct from the origin. This is called Huygens' principle.

One interesting aspect of Huygens' principle is that it is false in even dimensions, for example in two dimensions. This is a matter of common experience. After you toss a pebble into a pond, the resulting wave is not simply an expanding circular ring with still water inside and outside the ring. Rather, the water inside the expanding ring continues to ripple, with decreasing amplitude, as time progresses.

How is this reflected in the mathematical formula for the solution? Here is the formula in two dimensions.

$$u(\mathbf{x},t) = \frac{1}{2\pi c}\iint_{\mathbf{y}:|\mathbf{x}-\mathbf{y}|\le ct} \frac{u_0(\mathbf{y})}{\sqrt{c^2t^2 - |\mathbf{x}-\mathbf{y}|^2}} d\mathbf{y} + \frac{\partial}{\partial t}\left[\frac{1}{2\pi c}\iint_{\mathbf{y}:|\mathbf{x}-\mathbf{y}|\le ct} \frac{v_0(\mathbf{y})}{\sqrt{c^2t^2 - |\mathbf{x}-\mathbf{y}|^2}} d\mathbf{y}\right]$$

Notice that the integral is now over the whole disk $\mathbf{y} : |\mathbf{x}-\mathbf{y}| \le ct$, not just the circle $\mathbf{y} : |\mathbf{x}-\mathbf{y}| = ct$. Suppose that at time $t = 0$ a pebble is thrown in the pond at $\mathbf{y} = (0,0)$. Then the integration region $\mathbf{y} : |\mathbf{x}-\mathbf{y}| \le ct$ will contain $(0,0)$ when ct is greater or equal to (and not just equal to) the distance from $\mathbf{x}$ to the origin. Thus there is an initial time when u becomes non-zero, and it may be non-zero at any time after that.

Problem 5.5: Suppose that at time $t = 0$ two tubas, located at $\mathbf{y} = (10,0,0)$ and at $\mathbf{y} = (-10,0,0)$ simultaneously let out a blast. For a given time t, describe the set of locations where a listener would hear both tubas simultaneously.

Sturm-Liouville problems

Introduction

I have often mentioned general eigenfunction expansions in this course, but the only real examples we have seen so far are Fourier series, sine and cosine series (which are really Fourier series too) and, in two dimensions, double sine or cosine series. We now want to consider a more general class of eigenvalue problems in one dimension.

Let $p(x)$, $q(x)$ and $r(x)$ be given real-valued functions on an interval, and let a_1, a_2, b_1 and b_2 be fixed real numbers, with a_1 and a_2 not both zero, and b_1 and b_2 not both zero. For convenience, we usually will assume that the interval is $[0,1]$. We wish to find the eigenfunctions $\phi(x)$ and eigenvalues λ satisfying the eigenvalue problem

$$-\frac{d}{dx}p(x)\frac{d}{dx}\phi + q(x)\phi = \lambda r(x)\phi \tag{6.1}$$

with boundary conditions

$$\begin{aligned} a_1\phi(0) + a_2\phi'(0) = 0 \\ b_1\phi(1) + b_2\phi'(1) = 0 \end{aligned} \tag{6.2}$$

Another way of writing the equation is

$$-(p(x)\phi')' + q(x)\phi = \lambda r(x)\phi$$

The boundary conditions we are allowing are called *separated* boundary conditions, because the conditions for the function at one end of the interval are separate from the conditions at the other end. Notice that sine and cosine series eigenfunctions fit into this framework. (For example, for the sine series $\phi_n(x) = \sin(n\pi x)$ with eigenvalues $\lambda_n = \pi^2 n^2$ we have $p(x) = 1$, $q(x) = 0$, $r(x) = 1$, $a_1 = 1$, $a_2 = 0$, $b_1 = 1$, $b_2 = 0$.) However, the normal Fourier series eigenfunctions (i.e., for functions periodic with period 1) don't fit into this framework, because the relevant boundary conditions would be periodic boundary conditions

$$\begin{aligned} \phi(0) = \phi(1) \\ \phi'(0) = \phi'(1) \end{aligned}$$

which are not of separated type.

So far we haven't made any assumptions about the functions $p(x)$, $q(x)$ and $r(x)$. The most usual assumptions are that $p(x)$ and $r(x)$ are strictly positive functions: $p(x) > 0$ and $r(x) > 0$ on the interval $[0,1]$. If this holds, the Sturm-Liouville problem is said to be *regular*. Regular Sturm-Liouville problems are the most well-behaved. On the other hand if $p(x)$ or $r(x)$ is zero for some x, perhaps at one of the endpoints, or if the interval under consideration is infinite (e.g., $[0,\infty)$ or $(-\infty,\infty)$ then the problem is said to be *singular*. Singular Sturm-Liouville problems do arise in practice. It is harder to make general statements about them, though. There are some phenomena (for example, continuous spectrum) which only can occur for singular Sturm-Liouville problems.

Some examples

Consider a one dimensional heat flow problem where the composition of the bar varies from point to point. In this case the thermal diffusivity α^2 is not constant but a function $\alpha^2(x)$. The heat equation then becomes

$$\frac{\partial}{\partial t}u(x,t) = \frac{\partial}{\partial x}\alpha^2(x)\frac{\partial}{\partial x}u(x,t)$$

with appropriate boundary conditions. For example, if the ends are insulated the boundary conditions would be

$$\frac{\partial}{\partial x}u(0,t) = \frac{\partial}{\partial x}u(1,t) = 0$$

To solve this heat conduction problem when α^2 was constant we used a cosine series. Now, however, we must use the eigenfunctions for the Sturm-Liouville eigenvalue problem

$$-\frac{d}{dx}\alpha^2(x)\frac{d}{dx}\phi = \lambda\phi$$

with boundary conditions

$$\phi'(0) = 0$$
$$\phi'(1) = 0$$

In other words $p(x) = \alpha^2(x)$, $q(x) = 0$, $r(x) = 1$, $a_1 = 0$, $a_2 = 1$, $b_1 = 0$ and $b_2 = 1$. This is a regular Sturm Liouville problem.

Now consider a heat conduction problem where $\alpha^2 = 1$ is constant, and one end is insulated, but at the other end the heat flowing out is proportional to the temperature at that end, so that $-\phi'(1) = \mu\phi(1)$ for some constant of proportionality μ. For this problem, the appropriate eigenfunctions will be solutions to

$$-\frac{d^2}{dx^2}\phi = \lambda\phi$$

with boundary conditions

$$\phi'(0) = 0$$
$$\mu\phi(1) + \phi'(1) = 0$$

This is a regular Sturm-Liouville problem with $p(x) = 1$, $q(x) = 0$, $r(x) = 1$, $a_1 = 0$, $a_2 = 1$, $b_1 = \mu$ and $b_2 = 1$.

A Sturm-Liouville eigenvalue problem also arises when we try to find the eigenfunctions for the Laplace operator on a disk of radius R, with Dirichlet boundary conditions. It is convenient to use polar co-ordinates. Then we are looking for eigenfunctions $\phi(r,\theta)$ and eigenvalues λ that satisisfy

$$-\Delta\phi = -\frac{\partial^2\phi}{\partial r^2} - \frac{1}{r}\frac{\partial\phi}{\partial r} - \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = \lambda\phi$$

with

$$\phi(R,\theta) = 0$$

We will try to find ϕ of the form $\phi(r,\theta) = f(r)e^{in\theta}$ (or, equivalently, of the form $\phi(r,\theta) = f(r)\cos(n\theta)$ and $f(r)\sin(n\theta)$) Such a ϕ will be an eigenfunction if

$$-f''(r)e^{in\theta} - \frac{1}{r}f'(r)e^{in\theta} + \frac{n^2}{r^2}f(r)e^{in\theta} = \lambda f(r)e^{in\theta}$$

The boundary condition will be satisfied when

$$f(R) = 0$$

Dividing the equation by $e^{in\theta}$, multiplying by r and rearranging the terms, we rewrite the equation as

$$-rf''(r) - f'(r) + \frac{n^2}{r}f(r) = \lambda rf(r)$$

This isn't quite the right form. But $rf''(r) + f'(r)$=$(rf')'$ so the equation can be written

$$-\frac{d}{dr}r\frac{d}{dr}f + \frac{n^2}{r}f(r) = \lambda rf$$

with boundary condition

$$f(R) = 0$$

This is an example of a singular Sturm-Liouville problem on the interval $[0,R]$, because $p(r) = r$ vanishes at the left endpoint $r = 0$. Going along with this is the apparent absence of any boundary condition at $r = 0$. In fact, there is an implicit boundary condition at $r = 0$, namely, that the eigenfunction should stay bounded there.

Problem 6.1: Show that the eigenfunctions in the cosine series fit into the Sturm-Liouville framework.

Sturm-Liouville form

In the discussion above, we transformed the equation

$$-f''(r) - \frac{1}{r}f'(r) + \frac{n^2}{r^2}f(r) = \lambda f(r)$$

into standard Sturm Liouville form by multiplying by a suitable function and rewriting the terms involving derivatives. The reason we want to have our equations in this standard form, is that the orthogonality properties of the eigenfunctions are easy to read off from the standard form of the equation.

In fact, there is a general procedure for re-writing the eigenvalue problem

$$-P(x)\phi'' - Q(x)\phi' + R(x)\phi = \lambda\phi$$

in standard form. We assume here that $P(x) > 0$. The idea is to multiply the equation by a suitable integrating factor $\mu(x)$, so that the resulting equation

$$-\mu(x)P(x)\phi'' - \mu(x)Q(x)\phi' + \mu(x)R(x)\phi = \lambda\mu(x)\phi$$

has the desired form

$$-(\mu(x)P(x)\phi')' + \mu(x)R(x)\phi = \lambda\mu(x)\phi$$

Since $(\mu(x)P(x)\phi')' = \mu(x)P(x)\phi'' + (\mu'(x)P(x) + \mu(x)P'(x))\phi'$, the function μ that does the job satisfies the equation

$$\mu'(x)P(x) + \mu(x)P'(x) = \mu(x)Q(x)$$

This is a first order equation for μ which can be written

$$\frac{\mu'(x)}{\mu(x)} = \frac{Q(x)}{P(x)} - \frac{P'(x)}{P(x)}$$

Integrating, we get

$$\ln(\mu(x)) = \int^x \frac{Q(s)}{P(s)}ds - \ln(P(x)) + c$$

or

$$\mu(x) = \frac{C}{P(x)}e^{\int^x \frac{Q(s)}{P(s)}ds}$$

Clearly, we may set $C = 1$, since changing C just has the effect of multiplying the equation by an overall constant.

For example, the equation

$$-\phi'' + x^4\phi' = \lambda\phi$$

has $P(x) = 1$ and $Q(x) = -x^4$. It can be brought into standard form by multiplying it by

$$\mu(x) = e^{\int^x s^4 ds} = e^{-x^5/5}$$

The standard form is then

$$-(e^{-x^5/5}\phi')' = \lambda e^{-x^5/5}\phi$$

Problem 6.2: Bring the eigenvalue problem

$$-e^{2x}\phi'' - e^x\phi' + \sin(x)\phi = \lambda\phi$$

into standard form.

Main facts

Here are the main facts about the regular Sturm-Liouville eigenvalue problem.

Theorem 6.1 *The eigenvalues* λ *of a Sturm-Liouville problem are real.*

To see why this is true, we first allow the possibility of complex eigenfunctions and eigenvalues and define the inner product of two functions ϕ and ψ to be

$$\langle \phi, \psi \rangle = \int_0^1 \overline{\phi}(x)\psi(x)dx$$

We use the notation

$$L\phi = -\frac{d}{dx}p(x)\frac{d}{dx}\phi + q(x)\phi$$

Then, integrating by parts twice, we obtain

$$\begin{aligned}\langle \phi, L\psi \rangle &= -\int_0^1 \overline{\phi}(x)\Big(\frac{d}{dx}p(x)\frac{d}{dx}\psi(x)\Big)dx + \int_0^1 \overline{\phi}(x)q(x)\psi(x)dx \\ &= -\overline{\phi}(x)p(x)\psi'(x)\Big|_{x=0}^1 + \overline{\phi}'(x)p(x)\psi(x)\Big|_{x=0}^1 \\ &\qquad - \int_0^1 \Big(\frac{d}{dx}p(x)\frac{d}{dx}\overline{\phi}(x)\Big)\psi(x)dx + \int_0^1 \overline{\phi}(x)q(x)\psi(x)dx \\ &= p(1)(\overline{\phi}'(1)\psi(1) - \overline{\phi}(1)\psi'(1)) - p(0)(\overline{\phi}'(0)\psi(0) - \overline{\phi}(0)\psi'(0)) + \langle L\phi, \psi \rangle\end{aligned}$$

If ϕ and ψ obey the boundary conditions (6.2), then the boundary terms vanish. To see this, lets suppose a_1 is non-zero. Then $\phi(0) = -(a_2/a_1)\phi'(0)$ and $\psi(0) = -(a_2/a_1)\psi'(0)$. Taking the complex conjugate of the first of these equations, and recalling that a_2/a_1 is real, we also have $\overline{\phi}(0) = -(a_2/a_1)\overline{\phi}'(0)$. Thus

$$\overline{\phi}'(0)\psi(0) - \phi(0)\psi'(0) = -\overline{\phi}'(0)(a_2/a_1)\psi'(0) + (a_2/a_1)\overline{\phi}'(0)\psi'(0) = 0$$

So the first boundary term is zero. If a_1 happens to be zero, then a_2 cannot be zero, since they are not both zero. So we can use a similar argument. Similarly, the second boundary term is zero too.

Thus if ϕ and ψ obey the boundary conditions (6.2), then

$$\langle \phi, L\psi \rangle = \langle L\phi, \psi \rangle$$

In particular, if ϕ is an eigenfunction with eigenvalue λ, then

$$\lambda\langle \phi, r\phi \rangle = \langle \phi, \lambda r\phi \rangle = \langle \phi, L\phi \rangle = \langle L\phi, \phi \rangle = \langle \lambda r\phi, \phi \rangle = \overline{\lambda}\langle r\phi, \phi \rangle$$

But $\langle \phi, r\phi \rangle = \langle r\phi, \phi \rangle = \int_0^1 |\phi(x)|^2 r(x)dx$ is non-zero. Thus we can divide to obtain $\lambda = \overline{\lambda}$. In other words λ is real.

Once we know that λ is real, we can always find take the real part of the eigenvalue equation

$$L\phi = \lambda r\phi$$

obtaining

$$L\,\mathrm{Re}\,\phi = \lambda r\,\mathrm{Re}\,\phi$$

This shows that $\mathrm{Re}\,\phi$ is an eigenfunction too. Thus we may as well assume that the eigenfunctions are real.

Theorem 6.2 *The eigenfunctions ϕ_1 and ϕ_2 belonging to two distinct eigenvalues λ_1 and λ_2 are orthogonal in the inner product weighted by the function $r(x)$, i.e.,*

$$\int_0^1 \phi_1(x)\phi_2(x)r(x)dx = 0$$

This theorem follows from the calculation

$$\lambda_2 \int_0^1 \phi_1(x)\phi_2(x)r(x)dx = \langle \phi_1, \lambda_2 r\phi_2 \rangle = \langle \phi_1, L\phi_2 \rangle = \langle L\phi_1, \phi_2 \rangle = \lambda_1 \int_0^1 \phi_1(x)\phi_2(x)r(x)dx$$

So

$$(\lambda_2 - \lambda_1) \int_0^1 \phi_1(x)\phi_2(x)r(x)dx = 0$$

But $\lambda_2 - \lambda_1 \neq 0$, so we may divide by it to obtain the desired equation.

Theorem 6.3 *The eigenvalues form an increasing sequence $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$ with $\lambda_n \to \infty$ as $n \to \infty$. The corresponding eigenfunctions ϕ_1, ϕ_2, ... are simple (i.e, there is only one linearly independent eigenfunction).*

Finally, we have an expansion theorem. I won't state it in detail, since, as we have seen with Fourier series, its a little tricky. However the main idea is simple. More or less any function $f(x)$ can be expanded in a series

$$f(x) = \sum_{n=1}^{\infty} f_n \phi_n(x)$$

The coefficients are determined by multiplying by $r(x)\phi_m(x)$, integrating and exchanging the sum and the integral on the right. Using the orthogonality of the eigenfunctions this gives

$$\int_0^1 \phi_m(x)f(x)r(x)dx = \sum_{n=1}^{\infty} f_n \int_0^1 \phi_m(x)\phi_n(x)r(x)dx = f_m \int_0^1 \phi_m^2(x)r(x)dx$$

so that

$$f_m = \int_0^1 \phi_m(x)f(x)r(x)dx / \phi_m^2(x)r(x)dx$$

Example 1: mixed boundary conditions

In this section we consider the Sturm-Liouville eigenvalue problem

$$-\frac{d^2}{dx^2}\phi = \lambda\phi$$

with boundary conditions

$$\phi(0) = 0$$
$$\phi(1) + \phi'(1) = 0$$

This is a regular Sturm-Liouville problem with $p(x) = 1$, $q(x) = 0$, $r(x) = 1$, $a_1 = 1$, $a_2 = 0$, $b_1 = b_2 = 1$. To find the eigenvalues, we first consider the equation. This is a second order linear equation. It has two linearly independent solutions, namely $\cos(\mu x)$ and $\sin(\mu x)$ where $\mu^2 = \lambda$. Thus every possible solution of the equation can be written as a linear combination $\phi(x) = \alpha\cos(\mu x) + \beta\sin(\mu x)$. We need to decide which of these solutions satisfy the boundary conditions. The ones that do will be the eigenfunctions. The first boundary condition says

$$\phi(0) = \alpha\cos(\mu 0) + \beta\sin(\mu 0) = \alpha = 0.$$

Thus $\phi(x) = \beta\sin(\mu x)$, and since eigenfunctions are only defined up to a constant factor, we may as well set $\beta = 1$. The second boundary condition now reads

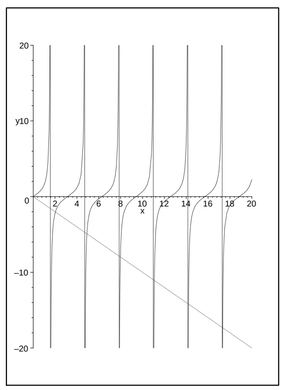
$$\phi(1) + \phi'(1) = \sin(\mu) + \mu\cos(\mu) = 0$$

This is an equation for μ. If μ solves this equation, then μ^2 is an eigenvalue and $\sin(\mu x)$ is an eigenfunction. Notice that since μ and $-\mu$ give rise to the same eigenvalue, we may as well assume that $\mu \ge 0$. Also, since $\mu = 0$ gives rise to the the solution $\phi(x) = 0$, which is not allowed as an eigenfunction, we may assume that $\mu > 0$.

Unfortunately, the equation for μ cannot be solved explicitly. Dividing by $\cos(\mu)$, it can be rewritten as

$$\tan(\mu) = -\mu$$

Thus the desired values of μ are the places where the graphs for $\tan(\mu)$ and $-\mu$ cross. If we draw the graphs, we can determine these values, at least approximately. Here is a picture of the two graphs.



From this picture we can see that there are infinitely many solutions $\mu_1, \mu_2, \ldots$. The first one is just above 2 Then, as n gets large μ_n gets closer and closer to the place where the tangent blows up, i.e., $\pi/2 + n\pi$. To get a more precise answer, we have to solve the equation numerically. Here are the first few

$$\begin{aligned}
\mu_1 &= 2.028757838\\
\mu_2 &= 4.913180439\\
\mu_3 &= 7.978665712\\
\mu_4 &= 11.08553841
\end{aligned}$$

So the first few eigenvalues and eigenfunctions are

$$\begin{aligned}
\lambda_1 &= \mu_1^2 = 4.115858365 & \phi_1(x) &= \sin(2.028757838 \cdot x)\\
\lambda_2 &= \mu_2^2 = 24.13934203 & \phi_2(x) &= \sin(4.913180439 \cdot x)\\
\lambda_3 &= \mu_3^2 = 63.65910654 & \phi_3(x) &= \sin(7.978665712 \cdot x)\\
\lambda_4 &= \mu_4^2 = 122.8891618 & \phi_4(x) &= \sin(11.08553841 \cdot x)
\end{aligned}$$

Lets verify the orthogonality relation. Since $r(x) = 1$, the general theory says that $\int_0^1 \phi_i(x)\phi_j(x)dx = 0$ for $i \ne j$. We can check this explicitly by doing the integral. A bit of calculation shows that for $\mu_i \ne \mu_j$

$$\int_0^1 \sin(\mu_i x)\sin(\mu_j x)dx = (\mu_i^2 - \mu_j^2)^{-1}\big(\mu_i\cos(\mu_i)\sin(\mu_j) - \mu_j\cos(\mu_j)\sin(\mu_i)\big)$$

But $\mu_i \cos(\mu_i) = -\sin(\mu_i)$ and $\mu_j \cos(\mu_j) = -\sin(\mu_j)$, so the expression simplifies to give zero.

Finally, lets use this Sturm-Liouville expansion to solve the following heat conduction problem.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$$\begin{aligned} u(0,t) &= 0 \\ u(1,t) + \frac{\partial u}{\partial x}(1,t) &= 0 \\ u(x,0) &= x \end{aligned}$$

To solve this, we expand $u(x,t)$ in a series of eigenfunctions for the Sturm-Liouville problem. Following the usual steps, we find

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\lambda_n t} \sin(\mu_n x)$$

where the b_n's are determined by the initial condition

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin(\mu_n x) = x$$

From the orthogonality relations, we then get

$$b_n = \int_0^1 x \sin(\mu_n x) dx \Big/ \int_0^1 \sin^2(\mu_n x) dx$$

A calculation shows

$$\int_0^1 \sin^2(\mu_n x) dx = \frac{-\cos(\mu_n)\sin(\mu_n) + \mu_n}{2\mu_n}$$

$$\int_0^1 x \sin(\mu_n x) dx = \frac{\sin(\mu_n) + \mu_n \cos(\mu_n)}{\mu_n^2}$$

Lets compute at least the leading term in this expansion. When $n = 1$ we obtain

$$b_n = 0.4358534909/0.5977353094 = 0.7291747435$$

so

$$u(x,t) \sim 0.7291747435 e^{-4.115858365t} \sin(2.028757838x)$$

Problem 6.3: Compute the leading term in the expansion of the solution to the heat conduction problem.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$$\begin{aligned} u(0,t) &= 0 \\ u(1,t) - \frac{\partial u}{\partial x}(1,t) &= 0 \\ u(x,0) &= 1 \end{aligned}$$

Review of the Constant coefficient and Euler equations

Two find the eigenfunctions of a Sturm-Liouville problem, we always start by finding two linearly independent solutions of a second order ODE. For a general second order linear ODE of the form

$$a(x)\phi''(x) + b(x)\phi'(x) + c(x)\phi(x) = 0$$

we can't write down the solutions explicitly. However, there are two common examples where we can write down explicit solutions.

The first is the constant coefficient equation

$$a\phi''(x) + b\phi'(x) + c\phi(x) = 0$$

In this case we try to find exponential solutions of the form $\phi(x) = e^{\lambda x}$. Substituting this guess into the equation we find that such an exponential is a solution of the polynomial equation

$$a\lambda^2 + b\lambda + c = 0$$

Most of the time, this equation has two solutions λ_1 and λ_2 given by the quadratic formula. We then obtain two linearly independent solutions $\phi_1 = e^{\lambda_1 x}$ and $\phi_2 = e^{\lambda_2 x}$. If the polynomial equation has a double root, i.e., the polynomial is of the form $a(\lambda - \lambda_1)^2$, then one solution is $\phi_1 = e^{\lambda_1 x}$, and the other (which can be found using reduction of order) is $\phi_2 = xe^{\lambda_1 x}$. If λ is complex, then we can write the resulting complex exponential solutions using sines and cosines.

Another equation we can always solve is the Euler equation. This equation has the form

$$ax^2\phi''(x) + bx\phi'(x) + c\phi(x) = 0$$

To solve this we try to find solutions of the form $\phi(x) = x^r$. (We assume that $x > 0$ so that x^r makes sense for any r). Substituting this guess into the equation we find that $\phi(x) = x^r$ is a solution if

$$ar(r-1) + br + c = ar^2 + (b-a)r + c = 0$$

Usually there are two distinct roots r_1 and r_2. In this case $\phi_1(x) = x^{r_1}$ and $\phi_2(x) = x^{r_2}$ are two linearly independent solutions. If the polynomial has a double root r_1, then the two independent solutions are $\phi_1(x) = x^{r_1}$ and $\phi_2(x) = \ln(x)x^{r_1}$.

Example 2

$$-\frac{d}{dx}x^2\frac{d}{dx}\phi = \lambda\phi$$

for $x \in [1, 2]$ with

$$\phi(1) = 0$$
$$\phi(2) = 0$$

Notice that this is a regular Sturm-Liouville problem on the interval $[1, 2]$ since x^2 is strictly positive there. The equation is an Euler equation that can be rewritten

$$-x^2\phi'' - 2x\phi' - \lambda\phi = 0$$

Substituting in $\phi(x) = x^r$ we find that this is a solution if

$$r(r-1) + 2r + \lambda = r^2 + r + \lambda = 0$$

or

$$r = \frac{-1 \pm \sqrt{1-4\lambda}}{2} = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - \lambda}$$

If $\lambda \neq 1/4$ there are two distinct roots, and the two linearly independent solutions are x^{r_1} and x^{r_2} with

$$r_1 = \frac{-1 + \sqrt{1-4\lambda}}{2} \quad r_2 = \frac{-1 - \sqrt{1-4\lambda}}{2}$$

If $\lambda = 1/4$ then there is a double root an the two solutions are x^{r_1} and $\ln(x)x^{r_1}$ with $r_1 = 1/2$.

Now we try to satisfy the boundary conditions with a linear combination of the two independent solutions.

If $\lambda = 1/4$ then the general solution is

$$\phi(x) = \alpha x^{r_1} + \beta \ln(x) x^{r_1}$$

with $r_1 = -1/2$. The left boundary conditions gives

$$\phi(1) = \alpha = 0$$

so $\phi(x) = \beta \ln(x) x^{r_1}$. Then the right boundary condition says

$$\beta \ln(2) 2^{r_1} = 0$$

which implies $\beta = 0$ too. So there is no non-zero eigenfunction for $\lambda = 1/4$, i.e., $1/4$ is not an eigenvalue.

If $\lambda \neq 1/4$ then the general solution is

$$\phi(x) = \alpha x^{r_1} + \beta x^{r_2}$$

with $r_1 = r_1(\lambda)$ and $r_2 = r_2(\lambda)$ given by the formulas above. The left boundary conditions gives

$$\phi(1) = \alpha + \beta = 0$$

so $\beta = -\alpha$ and $\phi(x) = \alpha(x^{r_1} - x^{r_2})$. We may as well set $\alpha = 1$. Then the other boundary conditions reads

$$\phi(2) = 2^{r_1} - 2^{r_2} = 0$$

or

$$2^{r_1 - r_2} = e^{(r_1 - r_2)\ln(2)} = 1$$

Thus

$$(r_1 - r_2)\ln(2) = 2\pi i n$$

for $n \in \mathbb{Z}$. This can be written

$$\sqrt{1-4\lambda} = 2\pi i n / \ln(2)$$

or

$$\sqrt{4\lambda - 1} = 2\pi n / \ln(2)$$

Taking the square and simplifying, we find that the eigenvalues are given by

$$\lambda_n = \frac{1}{4} + \frac{\pi^2 n^2}{\ln(2)^2}$$

for $n = 1, 2, 3, \ldots$. The corresponding eigenfunctions are

$$\begin{aligned} x^{r_1} - x^{r_2} &= x^{-1/2+i\pi n/\ln(2)} - x^{-1/2-i\pi n/\ln(2)} \\ &= x^{-1/2}\left(x^{i\pi n/\ln(2)} - x^{-i\pi n/\ln(2)}\right) \\ &= x^{-1/2}\left(e^{i\pi n\ln(x)/\ln(2)} - e^{-i\pi n\ln(x)/\ln(2)}\right) \\ &= 2ix^{-1/2}\sin(\pi n\ln(x)/\ln(2)) \end{aligned}$$

As always, we can drop a constant factor, in this case $2i$, since eigenfunctions are only defined up to a constant. Thus

$$\phi_n(x) = x^{-1/2}\sin(\pi n\ln(x)/\ln(2))$$

for $n = 1, 2, 3, \ldots$.

As an application, lets expand the function $f(x) = 1/\sqrt{x}$ in an eigenfunction expansion. We have that

$$f(x) = \sum_{n=1}^{\infty} f_n\phi_n(x)$$

with

$$f_n = \int_1^2 \phi_n(x)f(x)dx \Big/ \int_1^2 \phi_n^2(x)dx$$

We compute (using the change of variables $r = \ln(x)/\ln(2)$)

$$\begin{aligned} \int_1^2 \phi_n^2(x)dx &= \int_1^2 x^{-1}\sin^2(\pi n\ln(x)/\ln(2))dx \\ &= \ln(2)\int_0^1 \sin^2(\pi nr)dr \\ &= 2\ln(2) \end{aligned}$$

Similarly

$$\begin{aligned} \int_1^2 \phi_n(x)f(x)dx &= \int_0^1 x^{-1/2}\sin(\pi n\ln(x)/\ln(2))x^{-1/2}dx \\ &= \ln(2)\int_0^1\int_0^1 \sin(\pi nr)dr \\ &= \ln(2)(1-(-1)^n)/(n\pi) \end{aligned}$$

Thus

$$f_n = 2(1-(-1)^n)/(n\pi)$$

and

$$f(x) = x^{-1/2}\sum_{n=1}^{\infty}\frac{2(1-(-1)^n)}{n\pi}\sin(\pi n\ln(x)/\ln(2))$$

Problem 6.4: Solve the modified wave equation

$$\frac{\partial^2}{\partial t^2}u = \frac{\partial}{\partial x}x^2\frac{\partial}{\partial x}u$$

for $1 < x < 2$, with boundary conditions

$$u(1,t) = u(2,t) = 0$$

and initial conditions

$$u(x,0) = x^{-1/2}, \quad \frac{\partial}{\partial t}u(x,0) = 0$$

Example 3

Suppose we want to solve the modified heat equation

$$\frac{\partial}{\partial t}u = x^2\frac{\partial^2}{\partial x^2}u + 3x\frac{\partial}{\partial x}u$$

for $1 < x < 2$, with boundary conditions

$$u(1,t) = u(2,t) = 0$$

and some initial condition

$$u(x,0) = u_0(x)$$

The relevant eigenvalue problem is

$$-x^2\frac{d^2}{dx^2}\phi - 3x\frac{d}{dx}\phi = \lambda\phi.$$

with boundary condition

$$\phi(1) = \phi(2) = 0$$

However, this isn't in Sturm-Liouville form. So we first have to find the integrating factor μ. We can use the formula, but in this case its fairly easy to see by inspection that $\mu(x) = x$. Thus the Sturm-Liouville form of the equation is

$$-\frac{d}{dx}x^3\frac{d}{dx}\phi = \lambda x\phi$$

This tells us that the eigenfunctions will be orthogonal with respect to the weight function $r(x) = x$. To find the eigenvalues and eigenfunctions, its easier to go back to the original form of the eigenvalue equation. This is an Euler equation, and the analysis proceeds very similarly to the previous example. I'll omit the details. In the end we obtain

$$\lambda_n = 1 + \frac{n^2\pi^2}{\ln(2)^2}$$

for $n = 1, 2, 3, \ldots$ and

$$\phi_n(x) = x^{-1}\sin(n\pi\ln(x)/\ln(2))$$

Thus the solution to the heat equation will be

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} b_n e^{-\lambda_n t}\phi_n(x) \\ &= \sum_{n=1}^{\infty} b_n e^{-(1+\frac{n^2\pi^2}{\ln(2)^2})t} x^{-1}\sin(n\pi\ln(x)/\ln(2)) \end{aligned}$$

When computing the b_n's from the initial condition, we must remember to use the weight function $r(x) = x$. Thus

$$b_n = \int_1^2 u_0(x)\phi_n(x)x\,dx \Big/ \int_1^2 \phi_n^2(x)x\,dx$$

Problem 6.5: Fill in the missing details in the computation of λ_n and ϕ_n above.

Problem 6.6: Solve the heat equation

$$\frac{\partial}{\partial t}u = \frac{\partial^2}{\partial x^2}u + \frac{\partial}{\partial x}u$$

for $0 < x < 1$, with boundary conditions

$$u(0,t) = u(1,t) = 0$$

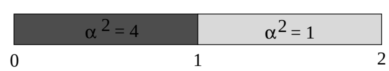
and initial condition

$$u(x,0) = 1$$

To make the problem a little less time consuming, you may use the information that all eigenvalues of the relevant Sturm-Liouville problem are greater than $1/4$.

Example 4

Suppose we join two metal rods together and consider the heat conduction problem for the joined rod. The diffusivity α^2 of the resulting rod will take two different values, with a jump where the two different metals are joined together. For definiteness, lets suppose that $\alpha^2 = 4$ for $0 \le x < 1$ and $\alpha^2 = 1$ for $1 < x \le 2$.



Suppose that the ends of the rod are kept at 0 degrees, and that at time $t = 0$ the initial heat distribution is given by $u_0(x)$. Then the heat conduction problem takes the form

$$\frac{\partial}{\partial t}u = \frac{\partial u}{\partial x}p(x)\frac{\partial u}{\partial x}u$$

where

$$p(x) = \begin{cases} 4 & \text{if } 0 \le x < 1 \\ 1 & \text{if } 1 < x \le 2 \end{cases}$$

The boundary conditions are

$$u(0,t) = u(2,t) = 0$$

and

$$u(x,0) = u_0(x).$$

Thus we are lead to the Sturm-Liouville eigenvalue problem

$$-\frac{d}{dx}p(x)\frac{d}{dx}\phi = \lambda\phi$$

with

$$\phi(0) = \phi(2) = 0.$$

However, this problem doesn't really make sense as it stands. The function $p(x)$ is not differentiable at $x = 1$, but has a jump discontinuity. So its not clear what $d/dxp(x)d/dx\phi$ means.

The equation does make sense on the subintervals $(0,1)$ and $(1,2)$. We interpret the equation to mean that $\phi(x)$ solves the equation for $x \in (0,1)$ (with $p(x) = 4$) and for $x \in (1,2)$ (with $p(x) = 1$). *In addition, we insist that $\phi(x)$ and its first derivative $\phi'(x)$ be continuous at $x = 1$.* In effect, this is like two extra boundary conditions in the middle of the rod.

With this interpretation, we can now find the eigenfunctions and eigenvalues. Let $\psi_1(x)$ denote the solution for $0 < x < 1$ and $\psi_2(x)$ denote the solution for $1 < x < 2$, so that

$$\phi(x) = \begin{cases} \psi_1(x) & \text{for } 0 < x < 1 \\ \psi_2(x) & \text{for } 1 < x < 2 \end{cases}$$

Then ψ_1 solves the equation

$$-4\psi_1''(x) = \lambda\psi_1(x)$$

This has solution

$$\psi_1(x) = a\cos(\mu x/2) + b\sin(\mu x/2)$$

where $\mu^2 = \lambda$. The function ψ_2 solves the equation

$$-\psi_1''(x) = \lambda\psi_1(x)$$

This has solution

$$\psi_2(x) = c\cos(\mu x) + d\sin(\mu x).$$

We now impose the boundary conditions and the matching conditions. The boundary condition $\phi(0) = \psi_1(0) = 0$ says

$$a = 0.$$

The boundary condition $\phi(2) = \psi_2(2) = 0$ says

$$c\cos(2\mu) + d\sin(2\mu) = 0$$

The matching condition $\psi_1(1) = \psi_2(1)$ says

$$a\cos(\mu/2) + b\sin(\mu/2) = c\cos(\mu) + d\sin(\mu).$$

Finally, the matching condition $\psi_1'(1) = \psi_2'(1)$ says

$$-a(\mu/2)\sin(\mu/2) + b(\mu/2)\cos(\mu/2) = -c\mu\sin(\mu) + d\mu\cos(\mu).$$

What we have are four linear equations in four unknowns. In fact, it is a homogeneous system of linear equations, which can be written as a matrix equation

$$A(\mu)\mathbf{x} = \mathbf{0}$$

where

$$A(\mu) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \cos(2\mu) & \sin(2\mu) \\ \cos(\mu/2) & \sin(\mu/2) & -\cos(\mu) & -\sin(\mu) \\ -(\mu/2)\sin(\mu/2) & (\mu/2)\cos(\mu/2) & \mu\sin(\mu) & -\mu\cos(\mu) \end{bmatrix}$$

and

$$\mathbf{x} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}.$$

Of course, this system of equations always has the solution $a = b = c = d = 0$. But this corresponds to $\phi(x) = 0$ which is not an eigenfunction. We are only interested in non-zero solutions to this equation. A non-zero solution will exist if the matrix is singular. Recall that a square matrix is singular if its determinant is zero. Thus, the values of μ for which a non-trivial solution $\mathbf{x}$ to the system of linear equations exists are those μ for which

$$\det(A(\mu)) = \mu\big(\sin(\mu/2)\cos(2\mu)\cos(\mu) + \sin(\mu/2)\sin(2\mu)\sin(\mu) \\ - \cos(\mu/2)\cos(2\mu)\sin(\mu)/2 + \cos(\mu/2)\sin(2\mu)\cos(\mu)/2\big) = 0$$

Even though there is a non-zero solution to $A(\mu)\mathbf{x} = \mathbf{0}$ when $\mu = 0$, this does not give rise to an eigenfunction. This is because the solution $\mathbf{x}$ has $a = c = 0$ so that $\psi_1(x) = b\sin(\mu x/2)$ and $\psi_2(x) = d\sin(\mu x)$. But these are both identically zero when $\mu = 0$.

Thus we may divide the determinant equation by μ and look for non-zero solutions. In principle, we should see if there are any purely imaginary solutions μ corresponding to negative eigenvalues. However, it would be impossible from the physical point of view if negative eigenvalues existed, since this would correspond to exponential growth of temperature for large time. Mathematically, one can show by integrating by parts that if λ and ϕ are an eigenvalue eigenfunction pair, then

$$\lambda = \int_0^2 p(x)\phi^2(x)dx / \int_0^2 \phi^2(x)dx.$$

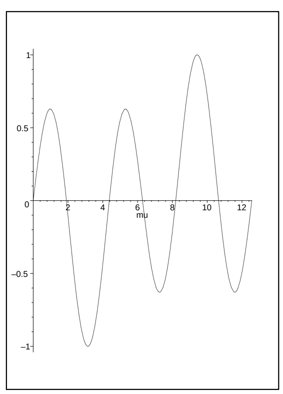
So λ must be positive and μ is therefore real.

Thus we must find the real solutions to

$$\sin(\mu/2)\cos(2\mu)\cos(\mu) + \sin(\mu/2)\sin(2\mu)\sin(\mu) - \cos(\mu/2)\cos(2\mu)\sin(\mu)/2 + \cos(\mu/2)\sin(2\mu)\cos(\mu)/2 = 0$$

Notice that the function on the right is periodic with period 4π. This means that if μ is a zero of this function, so is $\mu + 4n\pi$ for $n \in \mathbb{Z}$. In other words, we need only find the zeros between 0 and 4π. The rest of the zeros are then just shifted over by some multiple of 4π.

To find the zeros between 0 and 4π we proceed numerically. Here is a graph of the function.



Numerically, the zeros between 0 and 4π are determined to be

$$\begin{aligned} \mu_1 &\sim 1.910633236 \\ \mu_2 &\sim 4.372552071 \\ \mu_3 &\sim 6.283185307 \\ \mu_4 &\sim 8.193818543 \\ \mu_5 &\sim 10.65573738 \\ \mu_6 &\sim 12.56637061 \end{aligned}$$

So the first eigenvalue is $\lambda_1 = \mu_1^2 \sim 3.650519363$. To find the corresponding eigenfunction, we first must find the solution $\mathbf{x}$ to the system of linear equations $A(\mu_1)\mathbf{x} = 0$. I did this on a computer. The result is

$$\mathbf{x} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \sim \begin{bmatrix} 0 \\ 0.699062074 \\ -0.3805211946 \\ 0.4708709550 \end{bmatrix}$$

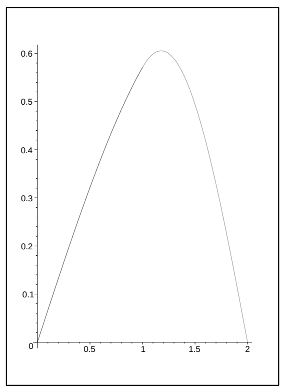
Thus the eigenfunction is equal to

$$b\sin(\mu_1 x/2) \sim (0.699062074) \cdot \sin((1.910633236) \cdot x/2)$$

for $0 \le x \le 1$ and equal to

$$c\cos(\mu_1 x) + d\sin(\mu_1 x) \sim (-0.3805211946) \cdot \cos((1.910633236) \cdot x) + (0.4708709550) \cdot \sin((1.910633236) \cdot x)$$

for $1 \le x \le 2$. Here is a picture of the eigenfunction.



Now we can go back and solve the original heat equation, at least approximately. The solution will be given by the standard eigenfunction formula

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\lambda_n t} \phi_n(x)$$

Where

$$b_n = \int_0^2 \phi_n(x) u_0(x) dx \Big/ \int_0^2 \phi_n^2(x) dx$$

Problem 6.7: Using the first term in the series as an approximation for $u(x,t)$, find the value of $u(0.5, 1)$ if $u_0(x) = 1$.

Problem 6.8: Describe in detail how you would find the eigenvalues and eigenfunctions for the Sturm-Liouville eigenvalue problem corresponding to insulated ends:

$$-\frac{d}{dx} p(x) \frac{d}{dx} \phi = \lambda \phi$$

with

$$\phi'(0) = \phi'(2) = 0.$$

Here $p(x)$ is the same discontinuous function as in the example.

Example 5

We now come to the singular Sturm-Liouville eigenvalue problems

$$-\frac{d}{dr}r\frac{d}{dr}f + \frac{n^2}{r}f = \lambda r f$$

with the boundary condition

$$f(R) = 0$$

and the implicit boundary condition that $f(r)$ remain bounded as $r \to 0$. This is a collection of infinitely many Sturm-Liouville problems, one for each n.

Let us recall the significance of these Sturm-Liouville eigenvalue problems. Fix $n \in \mathbb{Z}$, and let $\lambda_{n,m}$ and $f_{n,m}(r)$ for $m = 1, 2, \ldots$ be the eigenvalues and eigenfunctions of the Sturm-Liouville problem above. Then

$$\phi_{n,m}(r,\theta) = f_{n,m}(r)e^{in\theta}$$

are the eigenfunctions in polar co-ordinates of the Laplace operator on a disk of radius R.

To solve the equation we rewrite it as

$$r^2 f'' + r f' + (\mu^2 r^2 - n^2) f = 0$$

where $\mu^2 = \lambda$. Now define the function g by $g(r) = f(r/\mu)$, or, equivalently $f(r) = g(\mu r)$. Then $f'(r) = \mu g'(\mu r)$ and $f''(r) = \mu^2 g''(\mu r)$ so that the equation can be written

$$\mu^2 r^2 g''(\mu r) + \mu r g(\mu r) + (\mu^2 r^2 - n^2) g(\mu r) = 0$$

In other words, if we define the new variable $x = \mu r$ then g satisfies

$$x^2 g''(x) + x g'(x) + (x^2 - n^2) g(x) = 0 \tag{6.3}$$

This is a famous equation called Bessel's equation. For each value of n it has two linearly independent solutions called Bessel functions of order n, denoted $J_n(x)$ and $Y_n(x)$. Thus g must be a linear combination $g = aJ_n + bY_n$, and so

$$f(r) = aJ_n(\mu r) + bY_n(\mu r)$$

We must now see if we can satisfy the boundary conditions. To do this we need to know more about the Bessel functions $J_n(x)$ and $Y_n(x)$. The first fact we will use is that $J_n(x)$ is always bounded and $Y_n(x)$ is never bounded as $x \to 0$. Therefore, to satisfy the implicit boundary condition we must take $b = 0$. Then we might as well take $a = 1$ and write

$$f(r) = J_n(\mu r)$$

To satisfy the other boundary condition we need

$$f(R) = J_n(\mu R) = 0$$

In other words μR must be a zero of the Bessel function $J_n(x)$. So if $z_{n1}, z_{n,2}, z_{n,3}, \ldots$ are the zeros of $J_n(x)$, and

$$\mu_{n,m} = z_{n,m}/R,$$

then the eigenvalues for the Sturm-Liouville problem are

$$\lambda_{n,m}(r) = \mu_{n,m}^2 = z_{n,m}^2/R^2$$

and

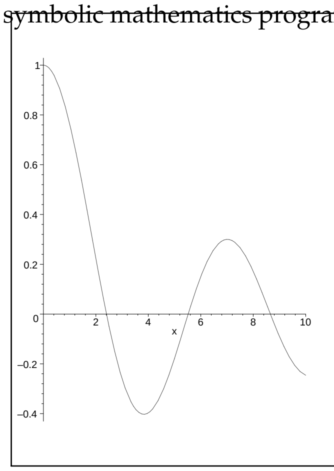
$$f_{n,m}(r) = J_n(\mu_{n,m} r)$$

The orthogonality relation for these eigenfuntions is

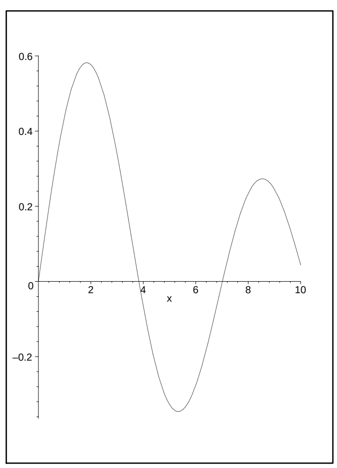
$$\int_0^R J_n(\mu_{n,m} r) J_n(\mu_{n,m'} r) r dr = 0$$

if $m \neq m'$.

How can we actually find the zeros of the Bessel functions $J_n(x)$? These zeros, at least for small n are tabulated in books, since they come up in many different problems. Nowadays, one can use the computer to compute them. The Bessel functions are built in to symbolic mathematics programs such as Maple. Here is a plot of $J_0(x)$



We can determine numerically that $z_{0,1} \sim 2.404825558$. Here is $J_1(x)$



We can determine numerically that $z_{1,1} \sim 3.831705970$.

Power series representation of $J_n(x)$

Lets examine the Bessel equation (6.3) for small x. When x is close to zero, then $x^2 - n^2$ is very close to n^2, so the equation is nearly an Euler equation

$$x^2 g''(x) + xg'(x) - n^2 g(x) = 0$$

This equation has the solutions x^n and x^{-n} when $n \neq 0$ and the solutions $x^0 = 1$ and $\ln(x)$ when $n = 0$. It is reasonable to suppose that there are solutions to the Bessel equation which look approximately like these functions for small x. This true, and in fact $J_n(x)$ is the solution that behaves like

$$J_n(x) \sim x^n \quad \text{as} \quad x \to 0$$

while $Y_n(x)$ behaves like

$$Y_n(x) \sim \begin{cases} \ln(x) & \text{if } n = 0 \\ x^{-n} & \text{if } n \neq 0 \end{cases} \quad \text{as} \quad x \to 0$$

To find a representation for $J_n(x)$ we will now consider a function of the form

$$g(x) = x^n(1 + a_1 x + a_2 x^2 + \cdots) = x^n \sum_{k=0}^{\infty} a_k x^k$$

and try to determine the coefficients a_n so that $g(x)$ solves the Bessel equation. We begin by computing

$$\begin{aligned} g'(x) &= nx^{n-1} \sum_{k=0}^{\infty} a_k x^k + x^n \left(\sum_{k=0}^{\infty} a_k x^k \right)' \\ &= nx^{n-1} \sum_{k=0}^{\infty} a_k x^k + x^n \sum_{k=0}^{\infty} k a_k x^{k-1} \end{aligned}$$

so

$$\begin{aligned} xg'(x) &= nx^n \sum_{k=0}^{\infty} a_k x^k + x^n \sum_{k=0}^{\infty} k a_k x^k \\ &= x^n \sum_{k=0}^{\infty} (n+k) a_k x^k \end{aligned}$$

Similarly

$$\begin{aligned} g''(x) &= n(n-1)x^{n-2} \sum_{k=0}^{\infty} a_k x^k + 2nx^{n-1} \left(\sum_{k=0}^{\infty} a_k x^k \right)' + x^n \left(\sum_{k=0}^{\infty} a_k x^k \right)'' \\ &= n(n-1)x^{n-2} \sum_{k=0}^{\infty} a_k x^k + 2nx^{n-1} \sum_{k=0}^{\infty} k a_k x^{k-1} + x^n \sum_{k=0}^{\infty} k(k-1) a_k x^{k-2} \end{aligned}$$

so

$$\begin{aligned} x^2 g''(x) &= n(n-1)x^n \sum_{k=0}^{\infty} a_k x^k + 2nx^n \sum_{k=0}^{\infty} k a_k x^k + x^n \sum_{k=0}^{\infty} k(k-1) a_k x^k \\ &= x^n \sum_{k=0}^{\infty} (n(n-1) + 2nk + k(k-1)) a_k x^k \\ &= x^n \sum_{k=0}^{\infty} ((n+k)^2 - (n+k)) a_k x^k \end{aligned}$$

We have

$$n^2 g(x) = x^n \sum_{k=0}^{\infty} n^2 a_k x^k$$

and finally

$$\begin{aligned} x^2 g(x) &= x^n \sum_{k=0}^{\infty} a_k x^{k+2} \\ &= x^n \sum_{k=2}^{\infty} a_{k-2} x^k \end{aligned}$$

Substituting all these expressions into the Bessel equation, we get

$$\begin{aligned} 0 &= x^n \left(\sum_{k=0}^{\infty} ((n+k)^2 - (n+k) + (n+k) - n^2) a_k x^k + \sum_{k=2}^{\infty} a_{k-2} x^k \right) \\ &= x^n \left(\sum_{k=0}^{\infty} ((n+k)^2 - n^2) a_k x^k + \sum_{k=2}^{\infty} a_{k-2} x^k \right) \end{aligned}$$

We may divide by x^n. Each coefficient in the resulting power series must be zero. The coefficient of x^k when $k = 0$ is

$$((n+0)^2 - n^2) a_0 = 0 \cdot a_0$$

This is zero no matter what a_0 is. So a_0 can be chosen arbitrarily. The coefficient of x^k when $k = 1$ is

$$((n+1)^2 - n^2) a_1 = (2n+1) a_1$$

Since n is always an integer, $(2n+1)$ is never zero. Therefore $a_1 = 0$. When $k \geq 2$ the second sum kicks in too. For $k = 2$ the coefficient of x^k is

$$((n+2)^2 - n^2) a_2 + a_0 = (4n+4) a_2 + a_0$$

This will be zero if

$$a_2 = \frac{-1}{4n+4} a_0$$

Similarly

$$a_3 = \frac{-1}{6n+9} a_1 = 0.$$

In general, for $k \geq 2$ we require

$$((n+k)^2 - n^2) a_k + a_{k-2} = (2n+k) k a_k + a_{k-2} = 0$$

This determines a_k in terms of a_{k-2}.

$$a_k = \frac{-1}{(2n+k)k} a_{k-2}$$

since the denominator $(2n+k)k$ is never zero. This equation is called a recursion relation. It implies that all the coefficients a_k for k odd are zero. It also lets us determine all the coefficients a_k for k even in terms of a_0. We get

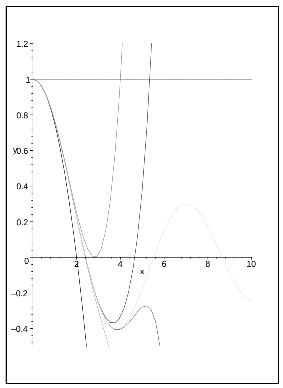
$$\begin{aligned} a_k &= \frac{-1}{(2n+k)k} a_{k-2} \\ &= \frac{1}{(2n+k)k(2n+k-2)(k-2)} a_{k-4} \\ &= \frac{-1}{(2n+k)k(2n+k-2)(k-2)(2n+k-4)(k-4)} a_{k-6} \\ &= \frac{(-1)^{k/2}}{(2n+k)(2n+k-2)(2n+k-4)\cdots(2n+2)\cdot k(k-2)(k-4)\cdots 2} a_0 \end{aligned}$$

Since these coefficients are decreasing rather rapidly as k gets large, the resulting power series converges to a solution $g(x)$ to the Bessel equation. If we choose $a_0 = 1$, the resulting solution is $J_n(x)$.

As an example, lets set $n = 0$ and compute $J_0(x)$. We have

$$J_0(x) = 1 - \frac{1}{2^2}x^2 + \frac{1}{(4\cdot 2)^2}x^4 - \frac{1}{(6\cdot 4\cdot 2)^2}x^6 + \cdots$$

In practice, one would want to use finitely many terms in this series. Here is a plot of the first term (that's the constant function 1), the first two terms, the first three terms and so on, together with the Bessel function $J_n(x)$.



The best approximation on this picture has terms up to $a_{10}x^{10}$. Notice that even this approximation is totally useless for computing the second zero, although it does a pretty good job on the first zero (2.404792604 compared to the true value of 2.404825558)

The disk in two dimensions

Using the eigenfunctions for the Laplace operator on the disk with Dirichlet boundary conditions, we can now solve the heat equation, the wave equation and the Poisson equation (i.e., Laplace's equation with an inhomogeneous term). For a disk of radius R, the eigenvalues and eigenfunctions are

$$\lambda_{n,m} = \mu_{n,m}^2$$

$$\phi_{n,m}(r,\theta) = J_n(\mu_{n,m}r)e^{in\theta}$$

where $R\mu_{n,m}$ is the mth zero on the positive real axis of the Bessel function $J_n(r)$.

These are complex eigenfunctions, so we must remember to put in the complex conjugate in the orthogonality relations. The Bessel functions appearing are real valued, hence $\overline{\phi}_{n,m}(r,\theta) = J_n(\mu_{n,m}r)e^{-in\theta}$. The orthogonality relations for these eigenfunctions are

$$\int_0^{2\pi}\int_0^R \overline{\phi}_{n,m}(r,\theta)\phi_{n',m'}(r,\theta)rdrd\theta = \int_0^R J_n(\mu_{n,m}r)J_n(\mu_{n',m'}r)\int_0^{2\pi} e^{-in\theta}e^{in'\theta}d\theta rdr$$

Since

$$\int_0^{2\pi} e^{-in\theta}e^{in'\theta}d\theta = \begin{cases} 0 & \text{if } n \neq n' \\ 2\pi & \text{if } n = n' \end{cases}$$

we obtain

$$\int_0^{2\pi}\int_0^R \overline{\phi}_{n,m}(r,\theta)\phi_{n',m'}(r,\theta)rdrd\theta = \begin{cases} 0 & \text{if } n \neq n' \\ 2\pi\int_0^R J_n(\mu_{n,m}r)J_n(\mu_{n,m'}r)rdr & \text{if } n = n' \end{cases}$$
$$= \begin{cases} 0 & \text{if } n \neq n' \\ 0 & \text{if } n = n' \text{ but } m \neq m' \\ 2\pi\int_0^R J_n^2(\mu_{n,m}r)rdr & \text{if } n = n' \text{ and } m = m' \end{cases}$$

Here we used the orthogonality relation for the Bessel functions. This means that if $f(r,\theta)$ is an "arbitrary" function on the disk, written in polar co-ordinates, then f can be expanded

$$f(r,\theta) = \sum_{n=-\infty}^{\infty}\sum_{m=1}^{\infty} a_{n,m}\phi_{n,m}(r,\theta)$$
$$= \sum_{n=-\infty}^{\infty}\sum_{m=1}^{\infty} a_{n,m}J_n(\mu_{n,m}r)e^{in\theta}$$

with

$$a_{n,m} = \int_0^{2\pi}\int_0^R \overline{\phi}_{n,m}(r,\theta)f(r,\theta)rdrd\theta / \int_0^{2\pi}\int_0^R |\phi_{n,m}(r,\theta)|^2 rdrd\theta$$
$$= \int_0^{2\pi}\int_0^R J_n(\mu_{n,m}r)e^{-in\theta}f(r,\theta)rdrd\theta \Big/ \left(2\pi\int_0^R J_n^2(\mu_{n,m}r)rdr\right)$$

We can also find real eigenfunctions for the Laplace operator on the disk with Dirichlet boundary conditions. To see this, we first If we look at the Bessel equation, $x^2J'' + xJ' + (x^2 - n^2)J = 0$, we can see that it only depends on n^2. This means that $J_{-n}(r)$ solves the same equation as $J_n(r)$. If you accept the fact that there is only one solution to the Bessel equation (up to multiplication by a constant) that stays bounded near zero then it must be that $J_{-n}(r)$ and $J_n(r)$ are multiples of each other. In fact, according to the standard definition of Bessel functions, the constants are chosen so that they are equal. Thus we have $J_{-n}(r) = J_n(r)$, so that $\mu_{-n,m} = \mu_{n,m}$ and $\lambda_{-n,m} = \lambda_{n,m}$. Another way of saying this is that for $n \neq 0$, $\lambda_{n,m}$ has multiplicity two, i.e., there are two independent eigenfunctions $\phi_{-n,m}$ and $\phi_{n,m}$ for the eigenvalue $\lambda_{n,m}$. This means that the linear combinations $(\phi_{n,m} + \phi_{-n,m})/2$ and $(\phi_{n,m} - \phi_{-n,m})/2i$ will also be eigenfunctions with the same eigenvalue. The upshot is that we may also take and

$$\begin{aligned} \varphi_{n,m}(r,\theta) &= (\phi_{n,m}(r,\theta) + \phi_{-n,m}(r,\theta))/2 &&= J_n(\mu_{n,m}r)\cos(n\theta) \quad \text{for } n = 0,1,2,\ldots \\ \psi_{n,m}(r,\theta) &= (\phi_{n,m}(r,\theta) - \phi_{-n,m}(r,\theta))/2i &&= J_n(\mu_{n,m}r)\sin(n\theta) \quad \text{for } n = 1,2,\ldots \end{aligned}$$

to be our eigenfunctions. These eigenfunctions have the advantage that they are real valued. They lead to the expansion

$$f(r,\theta) = \sum_{m=1}^{\infty}\frac{a_{0,m}}{2}J_0(\mu_{0,m}r) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{n,m}J_n(\mu_{n,m}r)\cos(n\theta) + b_{n,m}J_n(\mu_{n,m}r)\sin(n\theta)$$

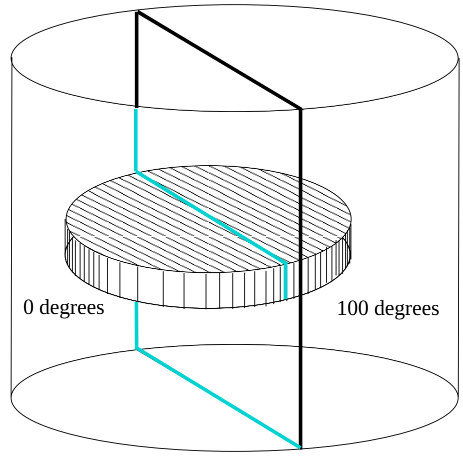
with

$$a_{n,m} = \int_0^{2\pi}\int_0^R J_n(\mu_{n,m}r)\cos(n\theta)f(r,\theta)rdrd\theta \Big/ \left(\pi\int_0^R J_n^2(\mu_{n,m}r)rdr\right)$$

and

$$b_{n,m} = \int_0^{2\pi}\int_0^R J_n(\mu_{n,m}r)\sin(n\theta)f(r,\theta)rdrd\theta \Big/ \left(\pi\int_0^R J_n^2(\mu_{n,m}r)rdr\right)$$

As our first example, lets go back to an heat conduction example we had before, except now with a circular plate of radius 1.



Suppose that the plate initially at 50° and then is placed in the divided beaker as shown. Lets calculate the temperature at later times. We will assume that the thermal diffusivity $\alpha^2 = 1$ in the units we are using.

The heat equation in polar co-ordinates is

$$\frac{\partial u}{\partial t} = \Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

The boundary conditions are

$$u(1,\theta,t) = \begin{cases} 100 & \text{if } 0 \le \theta < \pi \\ 0 & \text{if } \pi \le \theta < 2\pi \end{cases}$$

and the initial condition is

$$u(r,\theta,0) = 50$$

Since the boundary conditions are not homogeneous, we must begin by finding the steady state solution. The steady state solution $\phi(r,\theta)$ is the solution of Laplace's equation

$$\Delta\phi = 0$$

with boundary conditions

$$\phi(1,\theta) = \begin{cases} 100 & \text{if } 0 \le \theta < \pi \\ 0 & \text{if } \pi \le \theta < 2\pi \end{cases}$$

This is very similar to an example we did calculating the electrostatic potential in a cylinder. The solution is the solution of that example multiplied by 100.

$$\begin{aligned} \phi(r,\theta) &= 50 + \sum_{\substack{n=-\infty \\ n \text{ odd}}}^{\infty} \frac{100}{\pi i n} r^{|n|} e^{in\theta} \\ &= 50 + \sum_{n=0}^{\infty} \frac{200}{\pi(2n+1)} r^{2n+1} \sin((2n+1)\theta) \end{aligned}$$

Now we write down the equation for $v(r,\theta,t) = u(r,\theta,t) - \phi(r,\theta)$, obtaining

$$\frac{\partial v}{\partial t} = \Delta v = \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}$$

with Dirichlet boundary conditions

$$v(1,\theta,t) = 0$$

and the initial condition

$$v(r,\theta,0) = 50 - \phi(r,\theta) = -\sum_{\substack{n=-\infty \\ n \text{ odd}}}^{\infty} \frac{100}{\pi i n} r^{|n|} e^{in\theta}$$

The solution v can be written as an eigenfunction expansion

$$v(r,\theta,t) = \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} c_{n,m} e^{-\lambda_{n,m} t} \phi_{n,m}(r,\theta)$$

where $c_{n,m}$ are the coefficients in the expansion of the initial condition

$$\begin{aligned} c_{n,m} &= \int_0^1 \int_0^{2\pi} \overline{\phi}_{n,m}(r,\theta) v(r,\theta,0) r dr d\theta \Big/ \int_0^1 \int_0^{2\pi} |\phi_{n,m}(r,\theta)|^2 r dr d\theta \\ &= \int_0^{2\pi} \int_0^1 J_n(\mu_{n,m} r) e^{-in\theta} v(r,\theta,0) r dr d\theta \Big/ \left(2\pi \int_0^1 J_n^2(\mu_{n,m} r) r dr\right) \end{aligned}$$

Lets simplify the integral appearing in the numerator.

$$\begin{aligned} \int_0^1 \int_0^{2\pi} \overline{\phi}_{n,m}(r,\theta) v(r,\theta,0) r dr d\theta &= \sum_{\substack{n'=-\infty \\ n' \text{ odd}}}^{\infty} \int_0^1 \int_0^{2\pi} J_n(\mu_{n,m} r) e^{-in\theta} \frac{-100}{\pi i n'} r^{|n'|} e^{in'\theta} r dr d\theta \\ &= \sum_{\substack{n'=-\infty \\ n' \text{ odd}}}^{\infty} \frac{-100}{\pi i n'} \int_0^1 J_n(\mu_{n,m} r) r^{|n'|+1} dr \int_0^{2\pi} e^{i(n'-n)\theta} d\theta \\ &= \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{-200}{in} \int_0^1 J_n(\mu_{n,m} r) r^{|n|+1} dr & \text{if } n \text{ is odd} \end{cases} \end{aligned}$$

We have completed the calculation, but it is a good idea to put the final answer in a form where it is obvious that the temperature is real valued. To do this we combine

$$\begin{aligned} c_{n,m}\phi_{n,m} + c_{-n,m}\phi_{-n,m} &= \frac{-400 \int_0^1 J_n(\mu_{n,m} r') r'^{|n|+1} dr'}{2\pi \int_0^1 J_n^2(\mu_{n,m} r') r' dr'} J_n(\mu_{n,m} r) \frac{e^{in\theta} - e^{-in\theta}}{2i} \\ &= \frac{-200 \int_0^1 J_n(\mu_{n,m} r') r'^{|n|+1} dr'}{\pi \int_0^1 J_n^2(\mu_{n,m} r') r' dr'} \sin(n\theta) \end{aligned}$$

Thus

$$\begin{aligned} v(r,\theta,t) &= \sum_{\substack{n=1 \\ n \text{odd}}}^{\infty} \sum_{m=1}^{\infty} e^{-\lambda_{n,m} t} \frac{-200 \int_0^1 J_n(\mu_{n,m} r') r'^{|n|+1} dr'}{\pi \int_0^1 J_n^2(\mu_{n,m} r') r' dr'} \sin(n\theta) \\ &= \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} e^{-\lambda_{2n+1,m} t} \frac{-200 \int_0^1 J_{2n+1}(\mu_{2n+1,m} r') r'^{2n+2} dr'}{\pi \int_0^1 J_{2n+1}^2(\mu_{2n+1,m} r') r' dr'} \sin((2n+1)\theta) \end{aligned}$$

Finally,

$$u(r,\theta,t) = v(r,\theta,t) + \phi(r,\theta)$$

Problem 6.9: Solve the heat equation

$$\frac{\partial u}{\partial t} = \Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

with boundary conditions

$$u(1,\theta,t) = \begin{cases} 100 & \text{if } 0 \le \theta < \pi/2 \\ 0 & \text{otherwise} \end{cases}$$

and initial condition

$$u(r,\theta,0) = 0$$

Your answer may involve integrals of Bessel functions, as in the notes.

As our final example, we consider a vibrating circular membrane. The height of the membrane, in polar co-ordinates, satisfies the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \right)$$

with Dirichlet boundary conditions

$$u(1, \theta, t) = 0$$

and initial conditions

$$\begin{aligned} u(r, \theta, 0) &= u_0(r, \theta) \\ \frac{\partial u(r, \theta)}{\partial t} &= v_0(r, \theta) \end{aligned}$$

We can write down the solution as an eigenfunction expansion. This time we will use the real eigenfunctions. Since the result looks very complicated, lets review the steps of how we solve the wave equation with an eigenfunction expansion. For each fixed t, we expand the solution u in our eigenfuntions that satisfy the given homogeneous boundary conditions (in this case Dirichlet boundary conditions). This ensures that u satisfies the boundary condition too, as required. The coefficients depend on t.

$$u(r, \theta, t) = \sum_{m=1}^{\infty} \frac{a_{0,m}(t)}{2} J_0(\mu_{0,m} r) + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{n,m}(t) J_n(\mu_{n,m} r) \cos(n\theta) + b_{n,m}(t) J_n(\mu_{n,m} r) \sin(n\theta)$$

Now we substitute this expansion into the wave equation. We find that the equation is satisfied if the t dependent coefficients have the form

$$\begin{aligned} a_{0,m}(t) &= \left(\alpha_{n,m} \cos(c\sqrt{\lambda_{n,m}}\, t) + \beta_{n,m} \sin(c\sqrt{\lambda_{n,m}}\, t) \right) \\ b_{0,m}(t) &= \left(\gamma_{n,m} \cos(c\sqrt{\lambda_{n,m}}\, t) + \delta_{n,m} \sin(c\sqrt{\lambda_{n,m}}\, t) \right) \end{aligned}$$

Finally, we use the initial conditions to find $\alpha_{n,m}$, $\beta_{n,m}$, $\gamma_{n,m}$ and $\delta_{n,m}$. We have

$$u(r, \theta, 0) = \sum_{m=1}^{\infty} \frac{\alpha_{0,m}}{2} J_0(\mu_{0,m} r) + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \alpha_{n,m} J_n(\mu_{n,m} r) \cos(n\theta) + \gamma_{n,m} J_n(\mu_{n,m} r) \sin(n\theta) = u_0(r, \theta)$$

so that

$$\alpha_{n,m} = \int_0^{2\pi} \int_0^R J_n(\mu_{n,m} r) \cos(n\theta) u_0(r, \theta) r dr d\theta \Big/ \left(\pi \int_0^R J_n^2(\mu_{n,m} r) r dr \right)$$

and

$$\gamma_{n,m} = \int_0^{2\pi} \int_0^R J_n(\mu_{n,m} r) \sin(n\theta) u_0(r, \theta) r dr d\theta \Big/ \left(\pi \int_0^R J_n^2(\mu_{n,m} r) r dr \right)$$

And

$$\begin{aligned} \frac{\partial u(r, \theta)}{\partial t} = \sum_{m=1}^{\infty} & \frac{c\sqrt{\lambda_{n,m}}\, \beta_{0,m}}{2} J_0(\mu_{0,m} r) \\ & + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c\sqrt{\lambda_{n,m}}\, \beta_{n,m} J_n(\mu_{n,m} r) \cos(n\theta) + c\sqrt{\lambda_{n,m}}\, \delta_{n,m} J_n(\mu_{n,m} r) \sin(n\theta) = u_0(r, \theta) \end{aligned}$$

so that

$$\beta_{n,m} = \frac{1}{c\sqrt{\lambda_{n,m}}} \int_0^{2\pi} \int_0^R J_n(\mu_{n,m} r) \cos(n\theta) u_0(r, \theta) r dr d\theta \Big/ \left(\pi \int_0^R J_n^2(\mu_{n,m} r) r dr \right)$$

and

$$\delta_{n,m} = \frac{1}{c\sqrt{\lambda_{n,m}}} \int_0^{2\pi} \int_0^R J_n(\mu_{n,m} r) \sin(n\theta) u_0(r,\theta) r dr d\theta \Big/ \left(\pi \int_0^R J_n^2(\mu_{n,m} r) r dr \right)$$

Problem 6.10: Discuss how you can solve the Poisson equation

$$-\Delta u(r,\theta) = f(r,\theta)$$

on the disk of raduis 1, with boundary condition

$$u(1,\theta) = 0$$

Here f is a given function with expansion

$$f(r,\theta) = \sum_{n=-\infty}^{\infty} f_{n,m} \phi_{n,m}(r\theta)$$
