Realizing holonomic constraints

in classical and quantum mechanics

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Abstract

We consider the problem of constraining a particle to a smooth compact submanifold Σ of configuration space using a sequence of increasing potentials. We compare the classical and quantum versions of this procedure. This leads to new results in both cases: an unbounded energy theorem in the classical case, and a quantum averaging theorem. Our two step approach, consisting of an expansion in a dilation parameter, followed by averaging in normal directions, emphasizes the role of the normal bundle of Σ, and shows when the limiting phase space will be larger (or different) than expected.

1. Introduction

Consider a system of non-relativistic particles in a Euclidean configuration space $\mathbb{R}^{n+m}$ whose motion is governed by the Hamiltonian

$$H = \frac{1}{2}\langle p, p\rangle + V(x). \tag{1.1}$$

We are interested in the motion of these particles when their positions are constrained to lie on some n-dimensional smooth compact submanifold $\Sigma \subset \mathbb{R}^{n+m}$. In both classical and quantum mechanics there are accepted notions about what the constrained motion should be:

In classical mechanics, the Hamiltonian for the constrained motion is assumed to have the form (1.1), but whereas p and x originally denoted variables on the phase space $T^*\mathbb{R}^{n+m} = \mathbb{R}^{n+m} \times \mathbb{R}^{n+m}$, they now are variables on the cotangent bundle $T^*\Sigma$. The inner product $\langle p, p \rangle$ is now computed using the metric that Σ inherits from $\mathbb{R}^{n+m}$, and V now denotes the restriction of V to Σ.

In quantum mechanics, $\langle p, p \rangle$ is interpreted to mean $-\Delta$, where Δ is the Laplace operator, and $V(x)$ is the operator of multiplication by V. For unconstrained motion Δ is the Euclidean Laplacian on $\mathbb{R}^{n+m}$, and the Hamiltonian acts in $L^2(\mathbb{R}^{n+m})$. For constrained motion, the Laplace operator for Σ with the inherited metric is used, and the Hilbert space is $L^2(\Sigma, d\mathrm{vol})$.

In both cases the description of the constrained motion is intrinsic: it depends only on the Riemannian structure that Σ inherits from $\mathbb{R}^{n+m}$, but not on other details of the imbedding.

Of course, a constrained system of particles is an idealization. Instead of particles moving exactly on Σ, one might imagine there is a strong force pushing the particles onto the submanifold. The motion of the particles would then be governed by the Hamiltonian

$$H_\lambda = \frac{1}{2}\langle p, p \rangle + V(x) + \lambda^4 W(x) \tag{1.2}$$

where W is a positive potential vanishing exactly on Σ and λ is large. (The fourth power is just for notational convenience later on.) Does the motion described by H_λ converge to the intrinsic constrained motion as λ tends to infinity? Surprisingly, the answer to this question depends on exactly how it is asked, and is often no.

A situation in classical mechanics where the answer is yes is described by Rubin and Ungar [RU]. An initial position on Σ and an initial velocity tangent to Σ are fixed. Then, for a sequence of λ's tending to infinity, the subsequent motions under H_λ are computed. As λ becomes large, these motions converge to the intrinsic constrained motion on Σ. This result is widely known, since it appears in Arnold's book [A1] on classical mechanics. However, from the physical point of view, it is neither completely natural to require that the initial position lies exactly on Σ, nor that the initial velocity be exactly tangent. Rubin and Ungar also consider what happens if the initial velocity has a component in the direction normal to Σ. In this case, the motion in the normal direction is highly oscillatory, and there is an extra potential term, depending on the initial condition, in the Hamiltonian for the limiting motion on Σ. In their proof, Σ is assumed to have co-dimension one. A more complete result is given by Takens [T]. Here the initial conditions are allowed to depend on λ in such a way that the initial position converges to a point on Σ and the initial energy remains bounded. (We will give precise assumptions below.) Once again, the limiting motion on Σ is governed by a Hamiltonian with an additional potential. Takens noticed that a non-resonance condition on the eigenvalues of the Hessian of the constraining potential W along Σ is required to prove convergence. He also gave an example showing that if the Hessian of W has an eigenvalue crossing, so that the non-resonance condition is violated, then there may not be a good notion of limiting motion on Σ. In his example, he constructs two sequences of orbits, each one converging to an orbit on Σ. These limiting orbits are identical until they hit the point on Σ where the eigenvalues cross. After that, they are different. This means there is no differential equation on Σ governing the limiting motion. For other discussions of the question of realizing constraints see [A2] and [G]. A modern survey of the classical mechanical results that emphasizes the systematic use of weak convergence is given by Bornemann and Schütte [BS].

The quantum case was considered previously by Tolar [T], da Costa [dC1, dC2] and in the path integral literature (see Anderson and Driver [AD]). Related work can also be found in Helffer and Sjöstrand [HS1] [HS2], who obtained WKB expansions for the ground state, and in Duclos and Exner [DE], Figotin and Kuchment [FK], Schatzman [S] and Kuchment and Zeng [KZ]. The most general formal expansions appear in the preprint of Mitchell [M]. (We thank the referee for this reference.) There are really two aspects to the problem of realizing constraints: a large λ expansion followed by an averaging procedure to deal with highly oscillatory normal motion. Previous work in quantum mechanics concentrated on the first aspect (although a related averaging procedure for classical paths with a vanishingly small random perturbation can be found in [F] and [FW]). Already a formal large λ expansion reveals the interesting feature that the limiting Hamiltonian has an extra potential term depending on scalar and the mean curvatures. Since the mean curvature is not intrinsic, this potential does depend on the imbedding of Σ in $\mathbb{R}^{n+m}$.

It is not completely straightforward to formulate a theorem in the quantum case. We have chosen a formulation, modeled on the classical mechanical theorems, tracking a sequence of orbits with initial positions concentrating on Σ via dilations in the normal direction. Actually we consider the equivalent problem of tracking the evolution of a fixed vector governed by the Hamiltonian H_λ conjugated by unitary dilations. In order to obtain simple limiting asymptotics for the orbit we must assume that all the eigenvalues of the Hessian of the constraining potential W are constant on Σ. In fact we will assume that W is exactly quadratic. Our theorems show that for large λ the motion is approximated by the motion generated by an averaged limiting Hamiltonian $\overline{H}_B$, with superimposed normal oscillations generated by $\lambda^2 H_O$, where H_O is the normal harmonic oscillator Hamiltonian. The Hamiltonians $\overline{H}_B$ and H_O commute, so the motions are independent. These theorems do not require any non-resonance conditions on the eigenvalues of the Hessian of W. However, the limiting Hamiltonian $\overline{H}_B$ does not act in $L^2(\Sigma)$, but in $L^2(N\Sigma)$ where $N\Sigma$ is the normal bundle of Σ. It is only in certain situations where one can effectively ignore the motion in the normal directions and obtain a unitary group on $L^2(\Sigma)$ implementing the dynamics of the tangential motion. This occurs, for example, if (a) the eigenvalues of the Hessian of W are all distinct and non-resonant, (b) the normal bundle is trivial, and (c) we confine our attention to a simultaneous eigenspace of all the number operators for the normal motion. In the general situation, the dynamics of the additional degrees of freedom in $N\Sigma$ cannot be factored out, and we must be content with analysis on $L^2(N\Sigma)$.

Our formulation of the quantum theorems invites comparison with the classical mechanical results of Rubin and Ungar [RU] and Takens [T]. It turns out that extra potentials that appear in the two cases are quite different, and there is no obvious connection. Upon reflection, the reason for this difference is clear. If we have a sequence of initial quantum states whose position distribution is being squeezed to lie close to Σ, then by the uncertainty principle, the distribution of initial momenta will be spreading out, and thus the initial energy will be unbounded. However, the classical mechanical convergence theorems above all deal with bounded energies. The danger in considering unbounded energies is that even if the initial energy in the tangential mode is bounded, the coupling between tangential and normal modes may result in unbounded tangential energy in finite time. Our assumptions, which allow us to obtain a classical theorem despite the unbounded energy, are motivated by quantum mechanics. Our results for classical mechanics with unbounded initial energies are quite similar to our results in quantum mechanics.

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Section 2 contains a statement of the theorem of Rubin, Ungar and Takens on limiting orbits when the initial energies remain bounded. In Section 3 we state our expansion and averaging theorems in classical mechanics when the initial energies scale as they do in quantum mechanics. We also describe when the limiting motion can be thought of as a motion on Σ. These classical results are motivated by the parallel results in quantum mechanics, which we present in Section 4. The proofs of the theorems in Sections 3 and 4 are found in Sections 6 and 8 respectively, while Sections 5 and 7 contain background material needed in the proofs. This paper is an expanded and improved version of the announcement [FH].

2. Classical mechanics: bounded energy

To give a precise statement of our results we must introduce some notation. Let Σ be a smooth compact n-dimensional submanifold of $\mathbb{R}^{n+m}$. The normal bundle to Σ is the submanifold of $\mathbb{R}^{n+m} \times \mathbb{R}^{n+m}$ given by

$$N\Sigma = \{(\sigma, n) : \sigma \in \Sigma,\, n \in N_\sigma\Sigma\}$$

Here $N_\sigma\Sigma$ denotes the normal space to Σ at σ, identified with a subspace of $\mathbb{R}^{n+m}$.

There is a natural map from $N\Sigma$ into $\mathbb{R}^{n+m}$ given by

$$\iota : (\sigma, n) \mapsto \sigma + n.$$

We now fix a sufficiently small δ so that this map is a diffeomorphism of $N\Sigma_\delta = \{(\sigma, n) : \|n\| < \delta\}$ onto a tubular neighbourhood of Σ in $\mathbb{R}^{n+m}$. Then we can pull back the Euclidean metric from $\mathbb{R}^{n+m}$ to $N\Sigma_\delta$. Since we are interested in the motion close to Σ we may use $N\Sigma_\delta$ as the classical configuration space. This will be convenient in what follows, and is justified below.

We will want to decompose vectors in the cotangent spaces of $N\Sigma_\delta$ into horizontal and vertical vectors, so we now explain this decomposition. Let $\pi : N\Sigma \to \Sigma$ denote the projection of the normal bundle onto the base given by $\pi : (\sigma, n) \mapsto \sigma$. The vertical subspace of $T_{(\sigma,n)}N\Sigma$ is defined to be the kernel of $d\pi : T_{(\sigma,n)}N\Sigma \to T_\sigma\Sigma$. The horizontal subspace is then defined to be the orthogonal complement (in the pulled back metric) of the vertical subspace. Using the identification of $T_{(\sigma,n)}N\Sigma$ with $T^*_{(\sigma,n)}N\Sigma$ given by the metric we obtain a decomposition of cotangent vectors into horizontal and vertical components as well. We will denote by (ξ, η) the horizontal and vertical components of a vector in $T^*_{(\sigma,n)}N\Sigma$.

The decomposition can be explained more concretely as follows. For each point $\sigma \in \Sigma$, we may decompose $T_\sigma\mathbb{R}^{n+m} = T_\sigma\Sigma \oplus N_\sigma\Sigma$ into the tangent and normal space. Using the natural identification of all tangent spaces with $\mathbb{R}^{n+m}$, we may regard this as a decomposition of $\mathbb{R}^{n+m}$. Let P_σ^T and P_σ^N be the corresponding orthogonal projections. Since we are thinking of $N\Sigma$ as an $n+m$–dimensional submanifold of $\mathbb{R}^{n+m} \times \mathbb{R}^{n+m}$, we can identify $T_{(\sigma,n)}N\Sigma$ with the $n+m$–dimensional subspace of $\mathbb{R}^{n+m} \times \mathbb{R}^{n+m}$ given by all vectors of the form $(X, Y) = (\dot\sigma(0), \dot n(0))$, where $(\sigma(t), n(t))$ is a curve in $N\Sigma$ passing through (σ, n) at time $t = 0$. The inner product of two such tangent vectors is

$$\langle (X_1, Y_1), (X_2, Y_2)\rangle = \langle X_1 + Y_1, X_2 + Y_2\rangle \tag{2.1}$$

where the inner product on the right is the usual Euclidean inner product. For a tangent vector (X, Y), the decomposition into horizontal and vertical vectors is given by

$$(X, Y) = (X, P_\sigma^T Y) + (0, P_\sigma^N Y)$$

In the statements of our theorems we will want to express the fact that two cotangent vectors, for example $\xi_\lambda(t)$ and $\xi(t)$ in Theorem 2.1, are close, even though they belong to two different cotangent spaces. To do this we may use the imbedding to think of the vectors as elements of $\mathbb{R}^{2(n+m)}$. Then it makes sense to use the (Euclidean) norm of their difference, $\|\xi_\lambda(t) - \xi(t)\|$ to measure how close they are. We will use the symbol $\|\cdot\|$ in this situation, while $|\xi|$ will denote the norm of ξ as a cotangent vector.

We will assume that the constraining potential is a C^∞ function of the form

$$W(\sigma, n) = \frac{1}{2}\langle n, A(\sigma)n\rangle \tag{2.2}$$

where for each σ, $A(\sigma)$ is a positive definite linear transformation on $N_\sigma\Sigma$. The Hamiltonian (1.2) can then be written

$$H_\lambda(\sigma, n, \xi, \eta) = \frac{1}{2}\langle \xi, \xi\rangle + \frac{1}{2}\langle \eta, \eta\rangle + V(\sigma + n) + \lambda^4 W(\sigma, n) \tag{2.3}$$

Notice that on the boundary of $N\Sigma_{\delta_1}$, for $0 < \delta_1 < \delta$,

$$H_\lambda(\sigma, n, \xi, \eta) \geq c_1\lambda^4 - c_2$$

with

$$\begin{aligned} c_1 &= \inf_{(\sigma,n):\sigma\in\Sigma, \|n\|=\delta_1} W(\sigma + n) > 0 \\ c_2 &= \sup_{(\sigma,n):\sigma\in\Sigma, \|n\|=\delta_1} |V(\sigma + n)| \end{aligned}$$

By conservation of energy, this implies that an orbit under H_λ that starts out in $N\Sigma_{\delta_1}$ with initial energy less than $c_1\lambda^4 - c_2$ can never cross the boundary, and therefore stays in $N\Sigma_{\delta_1}$. We will only consider such orbits in this paper, and therefore are justified in taking our phase space to be $T^*N\Sigma_\delta$, or even $T^*N\Sigma$ if we extend H_λ in some arbitrary way.

Since we expect the motion in the normal directions to consist of rapid harmonic oscillations, it is natural to introduce action variables for this motion. There is one for each distinct eigenvalue $\omega_\alpha^2(\sigma)$ of $A(\sigma)$. Let $P_\alpha(\sigma)$ be the projection onto the eigenspace of $\omega_\alpha^2(\sigma)$. This projection is defined on $N_\sigma\Sigma$, which we may think of as the range of P_σ^N in $\mathbb{R}^{n+m}$. Thus the projection is defined on vertical vectors in $T_{(\sigma,n)}N\Sigma$ and, via the natural identification, on vertical vectors in $T^*_{(\sigma,n)}N\Sigma$. With this notation, the corresponding action variable, multiplied by λ^2 for notational convenience, is given by

$$I_\alpha^\lambda(\sigma, n, \xi, \eta) = \frac{1}{2\omega_\alpha(\sigma)}\langle \eta, P_\alpha\eta\rangle + \frac{\lambda^4\omega_\alpha(\sigma)}{2}\langle n, P_\alpha n\rangle \tag{2.4}$$

Notice that the total normal energy is given by $\sum_\alpha \omega_\alpha I_\alpha^\lambda$. The following is a version of the theorem of Takens and Rubin, Ungar.

Theorem 2.1 *Let Σ be a smooth compact n-dimensional submanifold of $\mathbb{R}^{n+m}$. Let the Hamiltonian H_λ on $T^*N\Sigma$ be given by (2.3) where $V, W \in C^\infty$, W has the form (2.2) and satisfies* *(i) The eigenvalues $\omega_\alpha^2(\sigma)$ of $A(\sigma)$ have constant multiplicity.*

Suppose that $(\sigma_\lambda, n_\lambda, \xi_\lambda, \eta_\lambda)$ are initial conditions in $T^*N\Sigma_\delta$ satisfying

(a) $\|\sigma_\lambda - \sigma_0\| + \|\xi_\lambda - \xi_0\| \to 0$,

(b) $I_\alpha^\lambda(\sigma_\lambda, n_\lambda, \xi_\lambda, \eta_\lambda) \to I_\alpha^0 > 0$,

as $\lambda \to \infty$. Let $(\sigma_\lambda(t), n_\lambda(t), \xi_\lambda(t), \eta_\lambda(t))$ denote the subsequent orbit in $T^*N\Sigma_\delta$ under the Hamiltonian H_λ. Suppose that $(\sigma(t), \xi(t))$ is the orbit in $T^*\Sigma$ with initial conditions (σ_0, ξ_0) governed by the Hamiltonian

$$h(\sigma, \xi) = \frac{1}{2}\langle \xi, \xi \rangle_\sigma + V(\sigma) + \sum_\alpha I_\alpha^0 \omega_\alpha(\sigma).$$

Then for any $T \geq 0$

$$\sup_{0 \leq t \leq T} \|\sigma_\lambda(t) - \sigma(t)\| + \|\xi_\lambda(t) - \xi(t)\| \to 0$$

as $\lambda \to \infty$.

Implicit in this statement is the fact that the approximating orbit stays in the tubular neighbourhood for $0 \leq t \leq T$, provided λ is sufficiently large. This theorem is actually true in greater generality. We can consider smooth constraining potentials W where $\frac{1}{2}\langle n, A(\sigma)n\rangle$ is the first term in an expansion. If we choose our tubular neighbourhood so that $W(\sigma + n) \geq c|n|^2$ and impose the non-resonance condition $\omega_\alpha(\sigma) \neq \omega_\beta(\sigma) + \omega_\gamma(\sigma)$ for every choice of α, β and γ and for every σ, then the same conclusion holds. This theorem is also really a local theorem: if we impose the conditions on W and the non-resonance condition locally, and take T to be a number less than the time where $\sigma(t)$ leaves the set where condition *(i)* is true, then the same conclusion holds as well.

Actually, Takens [T] only treats the case where all the eigenvalues ω_α are distinct and the normal bundle is trivial. On the other hand, he does not require that $I_\alpha^0 > 0$. This positivity is a technical requirement of our proof and arises because action angle co-ordinates are singular on the surface $I_\alpha^0 = 0$. Since Theorem 2.1 is a minor variation of known results, we will not give a proof here.

3. Classical mechanics: unbounded energy

We now describe our theorems in classical mechanics where the initial energies are diverging as they do in the quantum case. In quantum mechanics, the ground state energy of a harmonic oscillator $-\frac{1}{2}(d/dx)^2 + \frac{1}{2}\lambda^4\omega^2x^2$ is $\lambda^2\omega/2$. Thus we will assume that the initial values of the action variables I_α^λ scale like $\lambda^2 I_\alpha^0$, and therefore that the initial normal energy diverges like λ^2. Examining the effective Hamiltonian $h(\sigma, \xi)$ in Theorem 2.1, one would expect there to be a diverging $\lambda^2 \sum_\alpha I_\alpha^0 \omega_\alpha(\sigma)$ potential term similar to the constraining potential but with strength λ^2. If this potential is not constant, and thus has a local minimum (called a mini-well in [HS1, HS2]), no limiting orbit could be expected in general unless the initial positions were chosen to converge to such a minimum. For simplicity, we will assume that there are no mini-wells, i.e., the frequencies ω_α are constant.

The first step in our analysis is a large λ expansion. It is convenient to implement this expansion using dilations in the fibre of the normal bundle. It is also convenient to assume that our configuration space is all of $N\Sigma$. This makes no difference, since the orbits we are considering never leave $N\Sigma_\delta$.

The dilation $d_\lambda : N\Sigma \to N\Sigma$ is defined by

$$d_\lambda(\sigma, n) = (\sigma, \lambda n)$$

As with any diffeomorphism of the configuration space, d_λ has a symplectic lift D_λ to the cotangent bundle given by

$$D_\lambda = d_\lambda^{-1*} = d_{\lambda^{-1}}^*$$

The expression for D_λ in local co-ordinates is given by (5.1).

Instead of the original Hamiltonian H_λ we may now consider the equivalent pulled back Hamiltonian $L_\lambda = H_\lambda \circ D_\lambda^{-1}$. Since D_λ is a symplectic transformation, orbits under H_λ and orbits under L_λ are mapped to each other by D_λ and its inverse. Therefore, it suffices to study the dynamics of the scaled Hamiltonian L_λ.

A formal large λ expansion yields

$$L_\lambda = H_B + \lambda^2 H_O + O(\lambda^{-1})$$

where H_O is the harmonic oscillator Hamiltonian

$$H_O(\sigma, n, \xi, \eta) = \frac{1}{2}\langle \eta, \eta\rangle + \frac{1}{2}\langle n, A(\sigma) n\rangle \tag{3.1}$$

and H_B is the bundle Hamiltonian given by

$$H_B(\sigma, n, \xi, \eta) = \frac{1}{2}\langle J\xi, J\xi\rangle_\sigma + V(\sigma) \tag{3.2}$$

The inner product $\langle\cdot,\cdot\rangle_\sigma$ is the inner product on $T^*\Sigma$ defined by the imbedding. Here J denotes the identification of the horizontal subspace of $T^*_{(\sigma,n)}N\Sigma$ with the horizontal subspace of $T^*_\sigma\Sigma$ given in terms of the bundle projection map $\pi_{\sigma,n}$ by $J = d\pi_{\sigma,n}^{*-1}$. This map is well defined on the horizontal subspace, since $d\pi_{\sigma,n} : T_{(\sigma,n)}N\Sigma \to T_\sigma\Sigma$ is an isomorphism when restricted to the horizontal subspace of $T_{(\sigma,n)}N\Sigma$. Thus, its adjoint $d\pi^*_{\sigma,n}$ is an isomorphism of $T^*_\sigma\Sigma$ onto the horizontal subspace of $T^*_{(\sigma,n)}N\Sigma$. In local co-ordinates x_i, y_i defined in section 5 below, where x_i are co-ordinates for Σ, the map J simply identifies $dx_i \in T^*_{(\sigma,n)}N\Sigma$ with $dx_i \in T^*_\sigma\Sigma$.

Additional understanding of the Hamiltonians H_B and $H_O$can be obtained if we introduce another metric on $N\Sigma$. If $(X, Y) \in T_{(\sigma,n)}N\Sigma$, let

$$\langle (X,Y),(X,Y)\rangle_\lambda = \|X\|^2 + \lambda^{-2}\|P_\sigma^N Y\|^2. \tag{3.3}$$

(In Section 7 we describe in what sense this is a limiting form of the pulled-back, scaled, Euclidean metric.) If $\langle\cdot,\cdot\rangle^\lambda$ denotes the corresponding metric on the cotangent space, then

$$H_B + \lambda^2 H_O = \frac{1}{2}\langle(\xi,\eta),(\xi,\eta)\rangle^\lambda + \frac{\lambda^2}{2}\langle n, A(\sigma)n\rangle + V(\sigma)$$

The local co-ordinate expressions for H_B and H_O are given in (5.9) and (5.10).

We will use the notation ϕ_t^H to denote the Hamiltonian flow governed by the Hamiltonian H.

Theorem 3.1 *Let Σ be a smooth compact n-dimensional submanifold of $\mathbb{R}^{n+m}$. Let $L_\lambda = H_\lambda \circ D_\lambda^{-1}$, where the Hamiltonian H_λ on $T^*N\Sigma$ is given by (2.3). Assume that $V, W \in C^\infty$, W has the form (2.2), and that the eigenvalues ω_α^2 of $A(\sigma)$ do not depend on σ.*

Suppose that γ_λ are initial conditions in $T^*N\Sigma$ with $\gamma_\lambda \to \gamma_0$ as $\lambda \to \infty$.

Then for any $T \geq 0$

$$\sup_{0 \leq t \leq T} \left\| \phi_t^{L_\lambda}(\gamma_\lambda) - \phi_t^{H_B + \lambda^2 H_0}(\gamma_0) \right\| \to 0$$

as $\lambda \to \infty$.

In this theorem the normal energy of the initial conditions, $\lambda^2 H_O(\gamma_\lambda)$ grows like λ^2, since $H_O(\gamma_\lambda)$ is converging to $H_O(\gamma_0)$. This leads to increasingly rapid normal oscillations for both orbits $\phi_t^{L_\lambda}(\gamma_\lambda)$ and $\phi_t^{H_B+\lambda^2 H_0}(\gamma_0)$. Neither orbit converges as λ becomes large. It is only their difference that converges.

The convergence of the initial conditions is stated for the scaled variables γ_λ. To find out what this implies for the original variables $(\tilde{\sigma}_\lambda, \tilde{n}_\lambda, \tilde{\xi}_\lambda, \tilde{\eta}_\lambda) = D_\lambda^{-1}\gamma_\lambda$ we must determine the action of D_λ on horizontal and vertical vectors. This results in the following conditions

(a) $\tilde{\sigma}_\lambda \to \sigma_0$,

(b) $\lambda \tilde{n}_\lambda \to n_0$,

(c) $\tilde{\xi}_\lambda \to J\xi_0$, and

(d) $\lambda^{-1}\tilde{\eta}_\lambda \to \eta_0$

where $(\sigma_0, n_0, \xi_0, \eta_0) = \gamma_0$. Here we are thinking of σ, n as vectors in $\mathbb{R}^{n+m}$ and ξ, η as vectors in $\mathbb{R}^{2(n+m)}$. We may also compute what these conditions mean for the initial velocities $(X_\lambda, Y_\lambda) \in T_{(\tilde{\sigma}_\lambda, \tilde{n}_\lambda)}N\Sigma$, again thought of as vectors in $\mathbb{R}^{2(n+m)}$. It turns out that

(c′) $X_\lambda \to X_0$, and

(d′) $\lambda^{-1}Y_\lambda \to Y_0$.

This theorem gives a satisfactory description of the limiting motion if the Poisson bracket of H_B and H_O vanishes. Then the flows generated by H_B and H_O commute and the motion is given by the rapid oscillations generated by $\lambda^2 H_O$ superimposed on the flow generated by H_B. In this situation we can perform averaging by simply ignoring the oscillations.

An example where $\{H_B, H_O\}$ is zero is when Σ has codimension one, or, more generally, if the connection form, given by (3.10) below, vanishes. Then H_B only involves variables on $T^*\Sigma$, so the motion for large λ is a motion on Σ with independent oscillations in the normal variables. The Poisson bracket $\{H_B, H_O\}$ also vanishes if all the frequencies ω_α are equal, but in this case the motion generated by H_B need not only involve the variables on $T^*\Sigma$.

The motion generated by H_B can be thought of as a generalized minimal coupling type flow. (See [GS] for a description of the geometry of this sort of flow.) The flow has the property that the trajectories in $N\Sigma$ are parallel along their projections onto Σ. In particular, $|n|^2$ is preserved by this motion.

In general, when the frequencies are not all equal, the flows generated by H_B and $\lambda^2 H_O$ interact, and $H_B + \lambda^2 H_O$ generates a more complicated flow which need not be simply related to the flows generated by H_B and H_O. Let $\overline{H}_B$ be defined by

$$\overline{H}_B(\gamma) = \lim_{T\to\infty} T^{-1} \int_0^T H_B \circ \phi_t^{H_O}(\gamma) dt. \tag{3.4}$$

The existence of this limit follows from the Fourier expansion discussed below. This averaged Hamiltonian Poisson commutes with H_O. It turns out that the flow for large λ is the one generated by this Hamiltonian, with superimposed normal oscillations.

Theorem 3.2 *Assume that the assumptions of Theorem 3.1 hold. Let H_O, H_B and $\overline{H}_B$ be the Hamiltonians given by (3.1), (3.2) and (3.4) respectively. Let $\gamma_0 \in T^*N\Sigma$ and $T > 0$. Then*

$$\sup_{0\le t\le T} \left\| \phi_t^{H_B+\lambda^2 H_0}(\gamma_0) - \phi_t^{\lambda^2 H_0} \circ \phi_t^{\overline{H}_B}(\gamma_0) \right\| \to 0 \tag{3.5}$$

as $\lambda \to \infty$.

In this theorem we do not impose a non-resonance condition. However, the form of the averaged Hamiltonian $\overline{H}_B$ depends crucially on whether or not resonances are present.

To explain this further we introduce scaled action variables. Recall that the scaled Hamiltonian was defined by $L_\lambda = H_\lambda \circ D_\lambda^{-1}$. We perform a similar scaling on the action variables and define I_α by

$$I_\alpha^\lambda \circ D_\lambda^{-1} = \lambda^2 I_\alpha.$$

Then

$$I_\alpha(\sigma, n, \xi, \eta) = \frac{1}{2\omega_\alpha}\langle \eta, P_\alpha \eta\rangle + \frac{\omega_\alpha}{2}\langle n, P_\alpha n\rangle.$$

Suppose that there are m_0 distinct eigenvalues ω_α^2. Then the flows $\phi_t^{I_\alpha}$ are commuting harmonic oscillations in the normal variables. They are periodic, satisfying $\phi_{t+2\pi}^{I_\alpha} = \phi_t^{I_\alpha}$ We therefore obtain a group action Φ of the m_0 torus T^{m_0} on $T^*N\Sigma$ defined by

$$\Phi_\tau = \phi_{\tau_1}^{I_1} \circ \cdots \circ \phi_{\tau_{m_0}}^{I_{m_0}},$$

for $\tau = (\tau_1, \ldots, \tau_{m_0}) \in T^{m_0}$. Notice that $\phi_t^{H_O} = \Phi_{t\omega}$ where $\omega = (\omega_1, \ldots, \omega_{m_0})$.

Now we may perform a Fourier expansion of $H_B \circ \Phi_\tau$ yielding

$$H_B \circ \Phi_\tau = \sum_{\nu\in\mathbb{Z}^{m_0}} e^{i\langle \nu,\tau\rangle} F_\nu$$

so that

$$H_B \circ \phi_t^{H_O} = \sum_{\nu\in\mathbb{Z}^{m_0}} e^{it\langle \nu,\omega\rangle} F_\nu$$

It turns out that only finitely many F_ν's are non-zero. Thus we may exchange the integral and limit in the definition of $\overline{H}_B$ with the Fourier sum to obtain

$$\overline{H}_B = \sum_{\nu \in \mathbb{Z}^{m_0}} \left(\lim_{T \to \infty} T^{-1} \int_0^T e^{it\langle \nu, \omega \rangle} dt \right) F_\nu = \sum_{\nu \in \mathbb{Z}^{m_0} : \langle \nu, \omega \rangle = 0} F_\nu .$$

The non-resonance condition on the eigenvalues $\omega = (\omega_1, \ldots, \omega_{m_0})$ in this situation would be

$$\text{If } \nu \neq 0 \text{ and } F_\nu \neq 0 \text{ then } \langle \nu, \omega \rangle \neq 0. \tag{3.6}$$

If this condition holds, we find that $\overline{H}_B = F_0$.

We now examine the case $m_0 = m$, where there are m distinct frequencies ω_α. We wish to describe how the limiting motion generated by $\overline{H}_B$ can be thought of as taking place on Σ. To begin, since $\{\overline{H}_B, I_\alpha\} = 0$ for each α, each I_α is a constant of the motion, so the motion takes place on the level sets of $I_1, \ldots, I_m$. Furthermore, we want to to disregard the normal oscillations. Technically, we may do this by replacing the original phase space $T^*N\Sigma$, with its quotient by the group action Φ. This amounts to ignoring the angle variables in local action angle co-ordinates.

It turns out that

$$T^*N\Sigma/\Phi = T^*\Sigma \times \mathbb{R}^m, \tag{3.7}$$

where the variables in $\mathbb{R}^m$ are the action variables. Since these are constant, we may think of the motion as taking place on $T^*\Sigma$. To describe the identification (3.7) we first make a new direct sum decomposition of each cotangent space $T^*_{(\sigma,n)}N\Sigma$. Since there are m distinct eigenvalues $\omega_1, \ldots, \omega_m$, the corresponding eigenvectors, defined globally up to sign, give an orthonormal frame for the normal bundle. In this situation the co-ordinates $y_i = \langle n, n_i(\sigma) \rangle$ are also globally defined up to sign. Thus the subspace of $T^*_{(\sigma,n)}N\Sigma$ spanned by $dy_1, \ldots, dy_m$ is globally defined. This subspace is complementary to the horizontal subspace, but is not necessarily orthogonal. Given horizontal and vertical components (ξ, η) of a vector in $T^*_{(\sigma,n)}N\Sigma$, we may write $\xi + \eta = \xi_1 + \eta_1$ where ξ_1 is horizontal and η_1 is in the span of $dy_1, \ldots, dy_m$. The map from $T^*N\Sigma \to T^*\Sigma \times \mathbb{R}^m$ given by

$$(\sigma, n, \xi, \eta) \mapsto (\sigma, J\xi_1, I_1(\sigma, n, \xi, \eta), \ldots, I_m(\sigma, n, \xi, \eta))$$

is invariant under Φ and gives rise to the identification (3.7).

Now suppose that the values of $I_1, \ldots, I_m$ have been fixed by the initial condition. Then the Hamiltonian governing the motion on $T^*\Sigma$ depends on these "hidden" variables, and is given by

$$h_B(\sigma, \xi; I_1, \ldots, I_m) = \frac{1}{2}\langle \xi, \xi \rangle_\sigma + V(\sigma) + V_1(\sigma; I_1, \ldots, I_m), \tag{3.8}$$

provided the non-resonance condition holds. Given that the eigenvalues are distinct, the following implies (3.6)

$$\begin{aligned} &\text{If } j, k, l \text{ and } m \text{ are all distinct then } \omega_j + \omega_k \pm \omega_l - \omega_m \neq 0 \\ &\text{If } j, k \text{ and } l \text{ are all distinct then } 2\omega_j \pm \omega_k - \omega_l \neq 0 \end{aligned} \tag{3.9}$$

The extra potential V_1 is defined in terms of the frame for the normal bundle, $n_1(\sigma), \ldots, n_m(\sigma)$, consisting of normalized eigenvectors of $A(\sigma)$. Let $b_{k,l}$ be the associated connection one-form given by

$$b_{k,l}[\cdot] = \langle n_k, dn_l[\cdot]\rangle \tag{3.10}$$

Then

$$V_1(\sigma; I_1, \ldots, I_m) = \sum_{k,l} \frac{I_k I_l \omega_l}{\omega_k} |b_{k,l}|^2. \tag{3.11}$$

Notice that the norm $|b_{k,l}|$ is insensitive to the choice of signs for the frame.

4. Quantum mechanics

In quantum mechanics, we wish to understand the time evolution generated by H_λ for large λ, where H_λ is the Hamiltonian given by (1.2) with $\langle p, p\rangle = -\Delta$. As in the classical case, it is convenient to replace the original configuration space $\mathbb{R}^{n+m}$ with the normal bundle $N\Sigma$. We will show that if the initial conditions in $L^2(\mathbb{R}^{n+m})$ are supported near Σ then, to a good approximation for large λ, the time evolution stays near Σ. Thus we lose nothing by inserting Dirichlet boundary conditions on the boundary of the tubular neighbourhood of Σ, and may transfer our considerations to $L^2(N\Sigma_\delta, d\mathrm{vol})$, where $d$vol is computed using the pulled back metric. If we extend the pulled back metric, and make a suitable definition of H_λ in the complement of $N\Sigma_\delta$, we may remove the boundary condition. Thus we may assume that that the Hamiltonian H_λ acts in $L^2(N\Sigma, d\mathrm{vol})$.

More precisely, we let $g_{N\Sigma}$ be any complete smooth Riemannian metric on $N\Sigma$ that equals the metric induced from the imbedding in the region $\{(\sigma, n) : \|n\| < \epsilon\}$, for some $\epsilon < \delta$. For example, such a $g_{N\Sigma}$ could be obtained by smoothly joining the induced metric for small $\|n\|$ with the metric $\langle\cdot,\cdot\rangle_1$ given by (3.3) for large $\|n\|$. Let $d$vol denote the Riemannian density for $g_{N\Sigma}$. Let $V(\sigma, n)$ be a smooth bounded function on $N\Sigma$ such that $V(\sigma, n) = V(\sigma + n)$ when $\|n\| < \epsilon$. Our goal in this section is to analyze the time evolution generated by

$$H_\lambda = -\frac{1}{2}\Delta + V(\sigma, n) + \frac{\lambda^4}{2}\langle n, A(\sigma)n\rangle \tag{4.1}$$

acting in $L^2(N\Sigma, d\mathrm{vol})$. Here Δ denotes the Laplace-Beltrami operator for $g_{N\Sigma}$.

We now introduce the group of dilations in the normal directions by defining

$$(D_\lambda\psi)(\sigma, n) = \lambda^{m/2}\psi(\sigma, \lambda n).$$

This is a unitary operator from $L^2(N\Sigma, d\mathrm{vol}_\lambda)$ to $L^2(N\Sigma, d\mathrm{vol})$ where $d\mathrm{vol}_\lambda$ denotes the pulled back density $d\mathrm{vol}_\lambda(\sigma, n) = d\mathrm{vol}(\sigma, \lambda^{-1}n)$. Since the spaces $L^2(N\Sigma, d\mathrm{vol}_\lambda)$ depend on λ, and we want to deal with a fixed Hilbert space as $\lambda \to \infty$, we perform an an additional unitary transformation. Let

$$d\mathrm{vol}_{N\Sigma} = \lim_{\lambda\to\infty} d\mathrm{vol}_\lambda = d\mathrm{vol}_\Sigma \otimes d\mathrm{vol}_{\mathbb{R}^m}$$

Then the quotient of densities $d\text{vol}_{N\Sigma}/d\text{vol}_\lambda$ is a function on $N\Sigma$ and we may define M_λ to be the operator of multiplication by $\sqrt{d\text{vol}_{N\Sigma}/d\text{vol}_\lambda}$. The operator M_λ is unitary from $L^2(N\Sigma, d\text{vol}_{N\Sigma})$ to $L^2(N\Sigma, d\text{vol}_\lambda)$. Let

$$U_\lambda = D_\lambda M_\lambda. \tag{4.2}$$

Notice that the support of a family of initial conditions of the form $U_\lambda\psi$ is being squeezed close to Σ as $\lambda \to \infty$. We want to consider such a sequence of initial conditions. Therefore it is natural to consider the conjugated Hamiltonian

$$L_\lambda = U_\lambda^* H_\lambda U_\lambda,$$

since the evolution generated by L_λ acting on ψ is unitarily equivalent to the evolution generated by H_λ acting on $U_\lambda\psi$.

As a first step we perform a large λ expansion. Formally, this yields

$$L_\lambda = H_B + \lambda^2 H_O + O(\lambda^{-1})$$

where H_O is the quantum harmonic oscillator Hamiltonian in the normal variables, and H_B is quantum version of the corresponding classical Hamiltonian, except with an additional potential

$$K = \frac{n(n-1)}{4}s - \frac{n^2}{8}\|h\|^2.$$

Here s is the scalar curvature and h is the mean curvature vector (see equations (7.2) and (7.1)). Notice that this extra potential does depend on the imbedding of Σ in $\mathbb{R}^{n+m}$, since the mean curvature does. The quadratic forms for H_O and H_B are

$$\langle\psi, H_O\psi\rangle = \int_{N\Sigma} \frac{1}{2}\langle P^V d\psi, P^V d\psi\rangle_{\sigma,n} + \frac{1}{2}\langle n, A(\sigma)n\rangle|\psi|^2 d\text{vol}_{N\Sigma} \tag{4.3}$$

and

$$\langle\psi, H_B\psi\rangle = \int_{N\Sigma} \frac{1}{2}\langle JP^H d\psi, JP^H d\psi\rangle_\sigma + (V(\sigma,0) + K(\sigma))|\psi|^2 d\text{vol}_{N\Sigma}. \tag{4.4}$$

Local co-ordinate expressions for these operators are given by (7.7) and (7.6) below. As in the classical case, we can gain additional understanding of these operators by introducing the metric (3.3). Then

$$H_B + \lambda^2 H_O = -\frac{1}{2}\Delta_\lambda + \frac{\lambda^2}{2}\langle n, A(\sigma)n\rangle + V(\sigma,0) + K(\sigma),$$

where Δ_λ is the Laplace-Beltrami operator on $N\Sigma$ with the metric (3.3). Note that the volume element $d\text{vol}_{N\Sigma}$ is actually λ^m times the usual volume element associated to this metric (see Section 7).

The operator H_O is explicitly given on C^2 functions in its domain by the formula

$$(H_O\psi)(\sigma, n) = \left(-\frac{1}{2}\sum_{k=1}^m \frac{\partial^2}{\partial y_k^2} + \frac{1}{2}\langle n, A(\sigma)n\rangle\right)\psi(\sigma, \sum_{k=1}^m y_k n_k(\sigma)),$$

where $\{n_k(\sigma) : k = 1 \dots m\}$ is any orthonormal basis for $N\Sigma$ and $n = \sum_{k=1}^{m} y_k n_k(\sigma)$.

It is easy to show that with the metric (3.3), $N\Sigma$ is complete so that any positive integer power of $H_B + \lambda^2 H_O$ is essentially self-adjoint on C_0^∞ for $\lambda > 0$ [C]. Similarly, because H_O is basically a harmonic oscillator Hamiltonian, it is straightforward to show that any positive integer power of H_O is essentially self-adjoint on C_0^∞. The operator H_B is more complicated, but also can be shown to be essentially self-adjoint on C_0^∞. The argument is not difficult and will be omitted.

Theorem 4.1 *Let Σ be a smooth compact n-dimensional submanifold of $\mathbb{R}^{n+m}$. Let $g_{N\Sigma}$ be a complete smooth Riemannian metric on $N\Sigma$ that coincides with the induced metric when $\|n\| < \epsilon$, for some $\epsilon < \delta$, and suppose $V(\sigma, n)$ is a bounded smooth extension of $V(\sigma + n)$. Let H_λ be the Hamiltonian given by (4.1), acting in $L^2(N\Sigma, \mathrm{dvol})$. Assume that $A(\sigma)$ varies smoothly, and that the eigenvalues of ω_α^2 of $A(\sigma)$ do not depend on σ.*
Let $L_\lambda = U_\lambda^* H_\lambda U_\lambda$ acting in $L^2(N\Sigma, \mathrm{dvol}_{N\Sigma})$. Then, for every $\psi \in L^2(N\Sigma, \mathrm{dvol}_{N\Sigma})$ and every $T > 0$

$$\lim_{\lambda \to \infty} \sup_{0 \le t \le T} \left\| \left(e^{-itL_\lambda} - e^{-it(H_B + \lambda^2 H_O)} \right) \psi \right\| = 0$$

Just as in the classical case, this theorem provides a satisfactory description of the motion if $[H_B, H_O] = 0$, so that $\exp(-it(H_B + \lambda^2 H_O)) = \exp(-itH_B)\exp(-it\lambda^2 H_O)$. As before, this will happen, for example, if Σ has co-dimension one, or if all the frequencies ω_α are equal.

If Σ has co-dimension one, then the normal bundle is trivial. (We are assuming that Σ is compact.) Then we have $L^2(N\Sigma, \mathrm{dvol}_{N\Sigma}) = L^2(\Sigma, \mathrm{dvol}_\Sigma) \otimes L^2(\mathbb{R}, dy)$ and $H_B = h_B \otimes I$ for a Schrödinger operator h_B acting in $L^2(\Sigma, d\mathrm{vol}_\Sigma)$. Since $H_O = I \otimes h_O$ we have that $\exp(-it(H_B + \lambda^2 H_O)) = \exp(-ith_B) \otimes \exp(-it\lambda^2 h_O)$. This can be interpreted as a motion in $L^2(\Sigma, d\mathrm{vol}_\Sigma)$ with superimposed normal oscillations.

In the case where the frequencies ω_α are all equal, the normal bundle may be non-trivial, and there is not such a simple tensor product decomposition of $L^2(N\Sigma, d\mathrm{vol}_{N\Sigma})$. However, for some initial conditions ψ the limiting motion may again be thought of as taking place in $L^2(\Sigma, d\mathrm{vol}_\Sigma)$ with superimposed oscillations. For example, consider the subspace of functions in $L^2(N\Sigma, d\mathrm{vol}_{N\Sigma})$ that are radially symmetric in the fibre variable n. This subspace does have a tensor product decomposition $L^2(\Sigma, d\mathrm{vol}_\Sigma) \otimes L^2_{\mathrm{radial}}(\mathbb{R}^m, d^m y)$. It is an invariant subspace for H_B. Furthermore, the restriction of H_B to this subspace has the form $h_B \otimes I$. Thus, if ψ_0 is a radial function in n, then $\exp(-itL_\lambda)\psi_0 = \exp(-ith_B) \otimes \exp(-it\lambda^2 h_O)\psi_0$. As above, we interpret this as motion in $L^2(\Sigma, d\mathrm{vol}_\Sigma)$ with superimposed normal oscillations.

On the other hand, if the normal bundle is non-trivial, it may happen that the limiting motion takes place on a space of sections of a vector bundle over Σ. Instead of giving more details about the general case, we offer the following illustrative example. Instead of a normal bundle, consider the Möbius band $\mathcal{B}$ defined by $\mathbb{R} \times \mathbb{R} / \sim$, where $(x, y) \sim (x + 1, -y)$. This an $O(1)$ bundle over S^1 with fibre $\mathbb{R}$.

An L^2 function ψ on $\mathcal{B}$ can be thought of as a function on $\mathbb{R}\times\mathbb{R}$ satisfying $\psi(x+1,-y)=\psi(x,y)$. If we decompose $\psi(x,y)$, for fixed x, into odd and even functions of y

$$\psi(x,y)=\psi_{\text{even}}(x,y)+\psi_{\text{odd}}(x,y)$$

then $\psi_{\text{even}}(x+1,y)=\psi_{\text{even}}(x,y)$ and $\psi_{\text{odd}}(x+1,y)=-\psi_{\text{odd}}(x,y)$. (Notice that these are eigenfunctions for the left regular representation of $O(1)$ on $L^2(\mathbb{R})$.) Thus ψ_{even} can be thought of as an $L^2(\mathbb{R},dy)$ valued function on S^1, while ψ_{odd} can be thought of as an $L^2(\mathbb{R},dy)$ valued section of a line bundle over S^1 (which happens to be $\mathcal{B}$ itself). In this way we obtain the decomposition

$$L^2(\mathcal{B})=L^2(S^1,dx)\otimes L^2_{\text{even}}(\mathbb{R},dy)\,\oplus\,\Gamma(S^1,dx)\otimes L^2_{\text{odd}}(\mathbb{R},dy)$$

where Γ is the space of L^2 sections of $\mathcal{B}$.

In this example, the bundle is flat, so $H_B=-D_x^2+V(x)$ and $H_O=-D_y^2+y^2/2$ acting in $L^2(\mathcal{B},dxdy)$. Let $h_+=-D_x^2+V(x)$ acting in $L^2(S^1,dx)$ and $h_-=-D_x^2+V(x)$ acting in $\Gamma(S^1,dx)$. Let $h_0=-D_y^2+y^2/2$ acting in $L^2(\mathbb{R},dy)$, with $L^2_{\text{even}}(\mathbb{R},dy)$ and $L^2_{\text{odd}}(\mathbb{R},dy)$ as invariant subspaces. Then

$$e^{-it(H_B+\lambda^2H_O)}=e^{-ith_+}\otimes e^{-it\lambda^2h_O}\,\oplus\,e^{-ith_-}\otimes e^{-it\lambda^2h_O}$$

So if the initial condition happens to lie in $\Gamma\otimes L^2_{\text{odd}}$, then we would think of the limiting motion as taking place in Γ, with superimposed oscillations in L^2_{odd}.

When H_B and H_O do not commute, we perform a quantum version of averaging. Define $\overline{H}_B$ on C_0^∞ by

$$\overline{H}_B\psi=\lim_{T\to\infty}T^{-1}\int_0^\infty e^{itH_O}H_Be^{-itH_O}\psi\,dt \tag{4.5}$$

It can be shown that $\overline{H}_B$ is essentially self-adjoint.

Theorem 4.2 *Assume that the hypotheses of Theorem 4.1 hold. Let H_O, H_B, and $\overline{H}_B$ be the Hamiltonians defined by (4.3), (4.4) and (4.5). Then, for every $\psi\in L^2(N\Sigma,\mathrm{dvol}_{N\Sigma})$ and every $T>0$*

$$\lim_{\lambda\to\infty}\sup_{0\le t\le T}\left\|\left(e^{-it(H_B+\lambda^2H_O)}-e^{-it\lambda^2H_O}e^{-it\overline{H}_B}\right)\psi\right\|=0$$

The proof that this limit defining $\overline{H}_B$ exists parallels the discussion in classical mechanics. Suppose that there are m_0 distinct eigenvalues $\omega_1^2,\ldots,\omega_{m_0}^2$. For each $\alpha=1,\ldots,m_0$ define the operators I_α via the quadratic forms

$$\langle\psi,I_\alpha\psi\rangle=\int_{N\Sigma}\left(\frac{1}{2\omega_\alpha}\langle P^Vd\psi,P_\alpha P^Vd\psi\rangle+\frac{\omega_\alpha}{2}\langle n,P_\alpha n\rangle|\psi|^2\right)\mathrm{dvol}_{N\Sigma}$$

These operators all commute and satisfy

$$\sum_\alpha\omega_\alpha I_\alpha=H_O.$$

An expression for I_α in terms of local creation and annihilation operators will be given near the end of Section 7. In that section we will show that $e^{i\tau I_\alpha} H_B e^{-i\tau I_\alpha}$ is periodic in τ with period 2π. Thus if we conjugate H_B with $e^{i\sum \tau_\alpha I_\alpha}$, the resulting operator is defined on the torus T^{m_0} and has a Fourier expansion

$$e^{i\sum \tau_\alpha I_\alpha} H_B e^{-i\sum \tau_\alpha I_\alpha} = \sum_{\nu\in\mathbb{Z}^{m_0}} e^{i\langle \nu,\tau\rangle} F_\nu$$

Here $\tau = (\tau_1, \ldots, \tau_{m_0})$ and the coefficients F_ν are differential operators. As in the classical case, the sum is finite. Thus

$$e^{itH_O} H_B e^{-itH_O} = \sum_{\nu\in\mathbb{Z}^{m_0}} e^{it\langle \nu,\omega\rangle} F_\nu.$$

This shows that the limit defining $\overline{H}_B$ exists, and is given by

$$\overline{H}_B = \sum_{\nu\in\mathbb{Z}^{m_0}:\langle \nu,\omega\rangle=0} F_\nu.$$

As in the classical case, we may look for conditions under which the limiting motion can be considered to take place on Σ. Suppose that the eigenvalues $\omega_1, \ldots, \omega_m$ are all distinct, and, in addition, that the eigenvectors $n_k(\sigma)$ can be chosen to be smooth functions on all of Σ. Then the normal bundle is trivial, $N\Sigma = \Sigma\times\mathbb{R}^m$ and $L^2(N\Sigma, d\mathrm{vol}_{N\Sigma}) = L^2(\Sigma, d\mathrm{vol}_\Sigma)\otimes L^2(\mathbb{R}^m, d^m y)$. If the non-resonance condition (3.9) holds, then

$$\overline{H}_B = \left(-\frac{1}{2}\Delta_\Sigma + V(\sigma) + K(\sigma)\right)\otimes 1 + V_1.$$

The term V_1 is slightly different from (3.11), because terms arising in its computation do not all commute. It is given by

$$V_1 = \sum_{k,l}\left(\frac{I_k I_l \omega_l}{\omega_k} - \frac{1}{4}\right)|b_{k,l}|^2.$$

The joint eigenspaces of $I_1, \ldots, I_m$ are invariant subspaces for $\overline{H}_B$. The restriction of $\overline{H}_B$ to such a joint eigenspace is the Schrödinger operator $-\frac{1}{2}\Delta_\Sigma + V(\sigma) + K(\sigma) + \tilde{V}_1$, acting in $L^2(\Sigma, d\mathrm{vol}_\Sigma)$, where $\tilde{V}_1$ is obtained from V_1 by replacing the operators I_k by their respective eigenvalues. Thus $\overline{H}_B$ is a direct sum of Schrödinger operators acting in $L^2(\Sigma, d\mathrm{vol}_\Sigma)$.

5. Co-ordinate expressions

Our proofs will rely on local co-ordinate expressions for the quantities introduced above.

Suppose $x(\sigma)$ is a local co-ordinate map for Σ. Its inverse $\sigma(x)$ is a local imbedding of $\mathbb{R}^n$ onto $\Sigma\subset\mathbb{R}^{n+m}$. Given a local orthonormal frame $n_1(\sigma), \ldots, n_m(\sigma)$ for the normal bundle, we obtain local co-ordinates for $N\Sigma$ by setting

$$x_i(\sigma, n) = x_i(\sigma), \quad i = 1, \ldots, n$$

$$y_i(\sigma, n) = \langle n_i(\sigma), n\rangle, \quad i = 1, \ldots, m$$

We then may form the standard bases $\partial/\partial x_1, \ldots, \partial/\partial x_n, \partial/\partial y_1, \ldots, \partial/\partial y_m$ for the tangent spaces of $N\Sigma$ and $dx_1, \ldots, dx_n, dy_1, \ldots, dy_n$ for the cotangent spaces. This gives rise to local co-ordinates for $TN\Sigma$ and $T^*N\Sigma$ in the standard way. For the cotangent bundle, we will denote these by $(x, y, p, r) \in \mathbb{R}^{2(n+m)}$. Thus (x, y, p, r) denotes the cotangent vector $\sum p_i dx_i + \sum r_j dy_j$ in the cotangent space over $(\sigma(x), \sum_j y_j n_j(\sigma))$.

The standard symplectic form for $T^*N\Sigma$ is the two form given by

$$\omega = \sum_{i=1}^{n} dp_i \wedge dx_i + \sum_{j=1}^{m} dr_j \wedge dy_j$$

The dilation map D_λ is given in local co-ordinates by

$$D_\lambda(x, y, p, r) = (x, \lambda y, p, \lambda^{-1} r) \tag{5.1}$$

Clearly this map preserves the symplectic form ω.

We now compute the local expression for the metric. Let $\sigma_i(x) \in \mathbb{R}^{n+m}$ denote the vector $\partial\sigma(x)/\partial x_i$. The tangent vector $\partial/\partial x_i \in T_{(\sigma,n)}N\Sigma$ corresponds to the vector in $\mathbb{R}^{2(n+m)}$ given by $(\sigma_i, \sum_j y_j dn_j(\sigma)[\sigma_i])$. The tangent vector $\partial/\partial y_j$ corresponds to $(0, n_j(\sigma))$ Here $\sigma = \sigma(x)$, $\sigma_i = \sigma_i(x)$ and $n = \sum_j y_j n_j(\sigma(x))$. Using (2.1) for the inner product, we find that the local expression for the metric has block form

$$G(x, y) = \begin{bmatrix} G_\Sigma + C + BB^T & B \\ B^T & I \end{bmatrix} = \begin{bmatrix} I & B \\ 0 & I \end{bmatrix} \begin{bmatrix} G_\Sigma + C & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & B \\ 0 & I \end{bmatrix}^T \tag{5.2}$$

where $G_\Sigma = G_\Sigma(x)$ is the metric for Σ with matrix entries $\langle \sigma_i(x), \sigma_j(x) \rangle$, $B = B(x, y)$ is the matrix with entries

$$B_{i,j}(x, y) = \sum_k y_k \langle dn_k[\sigma_i], n_j \rangle \tag{5.3}$$

and where $C = C(x, y)$ is the matrix with entries

$$\begin{aligned} C_{i,j}(x, y) &= \sum_k y_k (\langle dn_k[\sigma_i], \sigma_j \rangle + \langle \sigma_i, dn_k[\sigma_j] \rangle) + \sum_{k,l} y_k y_l \langle dn_k[\sigma_i], dn_l[\sigma_j] \rangle - BB^T \\ &= \sum_k y_k (\langle dn_k[\sigma_i], \sigma_j \rangle + \langle \sigma_i, dn_k[\sigma_j] \rangle) + \sum_{k,l} y_k y_l \langle dn_k[\sigma_i], P_\sigma^T dn_l[\sigma_j] \rangle \end{aligned} \tag{5.4}$$

The geometrical meaning of the term $G_\Sigma + C$ is given in (7.12) below.

The inverse can be written

$$G^{-1}(x, y) = \begin{bmatrix} I & -B \\ 0 & I \end{bmatrix}^T \begin{bmatrix} (G_\Sigma + C)^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & -B \\ 0 & I \end{bmatrix} \tag{5.5}$$

The local expressions for the projections onto the vertical and horizontal subspaces can now be computed. Let P_V and P_H denote the projections for the tangent space and P^V and P^H the projections for the cotangent spaces. Then

$$P_V = \begin{bmatrix} 0 & 0 \\ B^T & I \end{bmatrix} \quad P_H = \begin{bmatrix} I & 0 \\ -B^T & 0 \end{bmatrix}$$

and

$$P^V = GP_VG^{-1} = \begin{bmatrix} 0 & B \\ 0 & I \end{bmatrix} \quad P^H = GP_HG^{-1} = \begin{bmatrix} I & -B \\ 0 & 0 \end{bmatrix}$$

Notice that the vertical subspace of $T_{(\sigma,n)}N\Sigma$ is the span of $\partial/\partial y_1, \ldots, \partial/\partial y_m$ and the horizontal subspace of $T^*_{\sigma,n}N\Sigma$ is the span of $dx_1, \ldots, dx_n$. The map $d\pi_{\sigma,n} : T_{(\sigma,n)}N\Sigma \to T_\sigma\Sigma$ sends $\partial/\partial x_i \in T_{(\sigma,n)}N\Sigma$ to $\partial/\partial x_i \in T_\sigma\Sigma$ and sends $\partial/\partial y_i \in T_{(\sigma,n)}N\Sigma$ to 0. From this it follows that $J = d\pi^{*-1}_{\sigma,n}$, defined on the horizontal subspace of $T^*_{\sigma,n}N\Sigma$ sends $dx_i \in T^*_{\sigma,n}N\Sigma$ to $dx_i \in T^*_\sigma\Sigma$. If (σ, n, ξ, η) has co-ordinates (x, y, p, r) then ξ has co-ordinates

$$P^H(x,y)\begin{bmatrix} p \\ r \end{bmatrix} = \begin{bmatrix} p - B(x,y)r \\ 0 \end{bmatrix}$$

so that $J\xi$ has co-ordinates

$$p - B(x,y)r.$$

We now compute the expressions for H_λ, H_B and H_O in local co-ordinates. We will abuse notation and use the same letters to denote functions on $T^*N\Sigma$ and their co-ordinate expressions. Suppose that the co-ordinates of (σ, n, ξ, η) are (x, y, p, r). Since

$$G^{-1}P^V = P^{VT}G^{-1}P^V = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \tag{5.6}$$

we have that

$$\langle \eta, \eta \rangle = \left\langle P^V \begin{bmatrix} p \\ r \end{bmatrix}, G^{-1}P^V \begin{bmatrix} p \\ r \end{bmatrix} \right\rangle = \langle r, r \rangle \tag{5.7}$$

Here, and in what follows, inner products involving co-ordinate vectors always refer to Euclidean inner products. For example, $\langle r, r \rangle = \sum_{i=1}^m r_i^2$. For the horizontal vectors, we have

$$\begin{bmatrix} I & -B \\ 0 & I \end{bmatrix} P^H = P^H$$

so that

$$\begin{aligned} \langle \xi, \xi \rangle &= \left\langle P^H \begin{bmatrix} p \\ r \end{bmatrix}, G^{-1}P^H \begin{bmatrix} p \\ r \end{bmatrix} \right\rangle \\ &= \langle (p - Br), (G_\Sigma + C)^{-1}(p - Br) \rangle \end{aligned} \tag{5.8}$$

Therefore the local co-ordinate expression for H_λ is

$$H_\lambda(x,y,p,r) = \frac{1}{2}\langle (p - Br), (G_\Sigma + C)^{-1}(p - Br) \rangle + \frac{1}{2}\langle r, r \rangle + \frac{\lambda^4}{2}\langle y, A(x)y \rangle + V(x,y)$$

Here $C = C(x,y)$ and $B = B(x,y)$ are the matrices appearing in the expression for the metric G, $A(x)$ is the matrix for $A(\sigma)$ in the basis given by the orthonormal frame $n_1, \ldots n_m$ used to define the co-ordinate system and $V(x,y) = V\big(\sigma(x) + \sum y_k n_k(\sigma(x))\big)$. Similarly

$$H_B(x,y,p,r) = \frac{1}{2}\langle (p - Br), G_\Sigma^{-1}(p - Br) \rangle + V(x,0) \tag{5.9}$$

where $B = B(x,y)$ and $G_\Sigma = G_\Sigma(x)$. Finally

$$H_O(x,y,p,r) = \frac{1}{2}\langle r,r\rangle + \frac{1}{2}\langle y, A(x)y\rangle \tag{5.10}$$

The expressions for H_O and I_α simplify if we can choose the vectors in the local orthonormal frame to be eigenvectors of $A(\sigma)$. This is always possible if there are no eigenvalue crossings. When, in addition, the eigenvalues $\omega_\alpha^2(\sigma)$ do not depend on σ there are further simplifications. In what follows we will assume that there are m_0 distinct constant eigenvalues ω_α^2 for $\alpha = 1,\ldots,m_0$, where ω_α^2 has multiplicity k_α. We will assume that the local orthonormal frame used to the define the co-ordinate system consists of eigenvectors for $A(\sigma)$. We label them $n_{\alpha,j}$, where $\alpha = 1,\ldots m_0$ and $j = 1,\ldots,k_\alpha$ where for each α, $n_{\alpha,j}$ is an eigenvector with eigenvalue ω_α^2. This means that the co-ordinates y and r now also acquire a double labelling.

First of all we have

$$H_O(x,y,p,r) = \frac{1}{2}\langle r,r\rangle + \frac{1}{2}\sum_\alpha \omega_\alpha^2 \sum_j y_{\alpha,j}^2$$

If the co-ordinates of (σ,n,ξ,η) are (x,y,p,r), then

$$\langle n, P_\alpha n\rangle = \sum_j y_{\alpha,j}^2$$

The vertical cotangent vector η has co-ordinates $P^V\begin{bmatrix} p \\ r\end{bmatrix}$. The corresponding tangent vector has co-ordinates $G^{-1}P^V\begin{bmatrix} p \\ r\end{bmatrix}$ which equals $\begin{bmatrix} 0 \\ r\end{bmatrix}$, by (5.6). Now the projection P_α, acting on tangent vectors, just picks off the basis vectors $\partial/\partial y_{\alpha,j}$, i.e., $P_\alpha \partial/\partial y_{\beta,j} = \delta_{\beta,\alpha}\partial/\partial y_{\beta,j}$. Thus

$$\langle \eta, P_\alpha \eta\rangle = \sum_j r_{\alpha,j}^2$$

Therefore

$$I_\alpha(x,y,p,r) = \frac{1}{2\omega_\alpha}\sum_j r_{\alpha,j}^2 + \frac{\omega_\alpha}{2}\sum_j y_{\alpha,j}^2$$

Notice that in this situation, where the vectors in the local orthonormal frame are eigenvectors of $A(\sigma)$, neither H_O nor I_α depend on x or p.

Now we introduce local action-angle co-ordinates. In analogy with creation and destruction operators in quantum mechanics, we define the complex quantities

$$a_{\alpha,j} = \frac{y_{\alpha,j}\omega_\alpha + i r_{\alpha,j}}{\sqrt{2\omega_\alpha}},$$

so that

$$\begin{aligned} y_{\alpha,j} &= \frac{1}{\sqrt{2\omega_\alpha}}(a_{\alpha,j} + a_{\alpha,j}^*) \\ r_{\alpha,j} &= -i\sqrt{\frac{\omega_\alpha}{2}}(a_{\alpha,j} - a_{\alpha,j}^*). \end{aligned}$$

The action variables $I_{\alpha,j} \in \mathbb{R}$ and angle variables $\varphi_{\alpha,j} \in S^1$ are then defined by

$$a_{\alpha,j} = \sqrt{I_{\alpha,j}} e^{i\varphi_{\alpha,j}}$$

Notice that $\sum_j I_{\alpha,j} = I_\alpha$. The change of co-ordinates from (x, y, p, r) to (x, φ, p, I) is symplectic, since $\sum dr_{\alpha,j} \wedge dy_{\alpha,j} = \sum dI_{\alpha,j} \wedge d\varphi_{\alpha,j}$. This makes it easy to compute the flow $\phi_t^{I_\alpha}$ in these co-ordinates. Hamilton's equations for the flow are

$$\dot{x}_i = 0, \quad \dot{p}_i = 0, \quad \dot{I}_{\alpha,j} = 0$$
$$\dot{\varphi}_{\alpha,j} = \delta_{\beta,\alpha}$$

Thus, under the flow $\phi_t^{I_\alpha}$ each $\varphi_{\alpha,j}$ is translated by t and all the other variables remain unchanged. This implies that under the group action $\Phi(\tau)$, with $\tau = (\tau_1, \ldots, \tau_{m_0})$ the quantities $a_{\alpha,j}$ evolve as $e^{-i\tau_\alpha} a_{\alpha,j}$.

We now compute the expression for H_B in action angle co-ordinates. We find

$$\begin{aligned}
(Br)_i &= \sum_{\alpha,j} B_{i,(\alpha,j)}(x,y) r_{\alpha,j} \\
&= \sum_{\beta,k,\alpha,j} b^i_{(\alpha,j),(\beta,k)}(x) r_{\alpha,j} y_{\beta,k} \\
&= \sum_{\beta,k,\alpha,j} \frac{b^i_{(\alpha,j),(\beta,k)}(x)}{2} (a_{\alpha,j} - a^*_{\alpha,j})(a_{\beta,k} + a^*_{\beta,k}) \sqrt{\frac{\omega_\alpha}{\omega_\beta}}
\end{aligned}$$

Here $b^i_{(\alpha,j),(\beta,k)}(x) = b_{(\alpha,j),(\beta,k)}[\sigma_i(x)]$ is the antisymmetric matrix given by (3.10). The expression for H_B is now obtained by substituting this formula for Br into (5.9), which we may rewrite as

$$H_B(x, p, \varphi, I) = \frac{1}{2} \sum_{i,l} p_i g^{i,l} p_l - \sum_{i,l} (Br)_i g^{i,l} p_l + \frac{1}{2} \sum_{i,l} (Br)_i g^{i,l} (Br)_l + V(x, 0)$$

Here $g^{i,l} = g^{i,l}(x)$ are the matrix elements of $G_\Sigma^{-1}(x)$. To obtain the expression for $H_B \circ \Phi(\tau)$ we simply replace each occurrence of $a_{\alpha,j}$ in the formula above with $e^{i\tau_\alpha} a_{\alpha,j}$. Since H_B contains only constant, quadratic and quartic terms in $a_{\alpha,j}, a^*_{\alpha,j}$, we see that the Fourier expansion of $H_B \circ \Phi(\tau)$ has finitely many terms, since the $\nu = (\nu_1, \ldots, \nu_{m_0})$'s that appear have $\sum_\alpha |\nu_\alpha| \in \{0, 2, 4\}$.

6. Proofs of theorems in classical mechanics

Proof of Theorem 3.1: We begin with some remarks about the co-ordinate charts for $T^*N\Sigma$. We will assume that the frames used to defined the co-ordinates consist of eigenvectors of $A(\sigma)$. We assume that each chart has the form $\{(\sigma, n, \xi, \eta) : \sigma \in \mathcal{U}, n \in N_\sigma\Sigma, \xi \in T^*_{\sigma,n}N\Sigma \text{ is horizontal}, \eta \in T^*_{\sigma,n}N\Sigma \text{ is vertical}\}$, where $\mathcal{U}$ is a co-ordinate chart for Σ. Since Σ is compact, there is an atlas with finitely many charts, and there exists a positive number ϵ_1 so that two points in $T^*N\Sigma$ both lie in a single chart if their projections onto Σ are a distance less than ϵ_1 apart.

We use the notation

$$\gamma_\lambda(t) = \phi_t^{L_\lambda}(\gamma_\lambda), \quad \gamma^\lambda(t) = \phi_t^{H_B+\lambda^2 H_0}(\gamma_0).$$

Our first estimates are large λ bounds on the components of

$$\gamma_\lambda(t) = (\sigma_\lambda(t), n_\lambda(t), \xi_\lambda(t), \eta_\lambda(t))$$

that follow from the conservation of energy. These bounds are

$$|n_\lambda(t)|, |\eta_\lambda(t)| \leq C \tag{6.1}$$

and

$$|\xi_\lambda(t)| \leq C\lambda \tag{6.2}$$

The analogous bounds also hold for $\gamma^\lambda(t) = (\sigma^\lambda(t), n^\lambda(t), \xi^\lambda(t), \eta^\lambda(t))$. Clearly $|n_\lambda(t)| = |y_\lambda(t)|$ and, by (5.7), $|\eta_\lambda(t)| = |r_\lambda(t)|$. Thus, (6.1) implies that $|y_\lambda(t)|$ and $|r_\lambda(t)|$ remain bounded.

To prove these we first consider the action of D_λ^{-1} on ξ_λ. Let $\gamma_\lambda = (\sigma_\lambda, n_\lambda, \xi_\lambda, \eta_\lambda)$ have co-ordinates $(x_\lambda, y_\lambda, p_\lambda, r_\lambda)$. Then $\xi_\lambda \in T^*_{\sigma_\lambda, n_\lambda}N\Sigma$ has co-ordinates

$$P^H \begin{bmatrix} p_\lambda \\ r_\lambda \end{bmatrix} = \begin{bmatrix} p_\lambda - B(x_\lambda, y_\lambda) r_\lambda \\ 0 \end{bmatrix}$$

We now wish to apply D_λ^{-1}. Since $B(x, y)$ is linear in y, the scaling in y_λ and in r_λ cancel. In other words

$$B(x_\lambda, \lambda^{-1} y_\lambda)\lambda r_\lambda = B(x_\lambda, y_\lambda) r_\lambda.$$

Thus $D_\lambda^{-1}\xi_\lambda \in T^*_{\sigma_\lambda, \lambda^{-1}n_\lambda}N\Sigma$ has the same co-ordinates as $\xi_\lambda \in T^*_{\sigma_\lambda, n_\lambda}N\Sigma$. This implies that as $\lambda \to \infty$,

$$\begin{aligned}
|D_\lambda^{-1}\xi_\lambda|^2 &= \left\langle \begin{bmatrix} p_\lambda - B(x_\lambda, y_\lambda) r_\lambda \\ 0 \end{bmatrix}, G^{-1}(x_\lambda, \lambda^{-1} y_\lambda) \begin{bmatrix} p_\lambda - B(x_\lambda, y_\lambda) r_\lambda \\ 0 \end{bmatrix} \right\rangle \\
&= \langle \big(p_\lambda - B(x_\lambda, y_\lambda) r_\lambda\big), \big(G_\Sigma(x_\lambda) + C(x_\lambda, \lambda^{-1} y_\lambda)\big)^{-1} \big(p_\lambda - B(x_\lambda, y_\lambda) r_\lambda\big) \rangle \\
&\to \langle \big(p_0 - B(x_0, y_0) r_0\big), G_\Sigma(x_0)^{-1} \big(p_0 - B(x_0, y_0) r_0\big) \rangle \\
&= |d\pi^{*-1}\xi_0|^2
\end{aligned} \tag{6.3}$$

Thus, for large λ, the initial energy satisfies

$$\begin{aligned}
L_\lambda(\gamma_\lambda) = H_\lambda \circ D_\lambda^{-1}(\gamma_\lambda) &\leq \frac{1}{2}|D_\lambda^{-1}\xi_\lambda|^2 + C_V + \frac{\lambda^2}{2}\left(|\eta_\lambda|^2 + \langle n_\lambda, A(\sigma_\lambda) n_\lambda \rangle\right) \\
&\leq C\lambda^2,
\end{aligned}$$

where C_V is an upper bound for V in a neighbourhood of Σ. Given this bound on the initial energies, we may assume that V is bounded, as was explained in the introduction. We now estimate the energy for later times t.

$$\begin{aligned} L_\lambda(\gamma_\lambda(t)) = H_\lambda \circ D_\lambda^{-1}(\gamma_\lambda(t)) &\geq \frac{1}{2}|D_\lambda^{-1}\xi_\lambda(t)|^2 - \|V\|_\infty + C\lambda^2\left(|\eta_\lambda(t)|^2 + |n_\lambda(t)|^2\right) \\ &\geq -\|V\|_\infty + C\lambda^2\left(|\eta_\lambda(t)|^2 + |n_\lambda(t)|^2\right) \end{aligned}$$

Since energy is conserved, i.e., $L_\lambda(\gamma_\lambda(t)) = L_\lambda(\gamma_\lambda)$, this implies (6.1). In a similar way we find that

$$|D_\lambda^{-1}\xi_\lambda(t)|^2 \leq C\lambda^2. \tag{6.4}$$

Now for $|y| < C_1$ sufficiently large λ there is a constant C such that

$$G^{-1}(x, y) < CG^{-1}(x, \lambda^{-1}y)$$

in any of the finitely many co-ordinate patches. Thus, (6.3) implies

$$|\xi_\lambda(t)| \leq |D_\lambda^{-1}\xi_\lambda(t)|,$$

so that (6.4) implies (6.2).

The proof of bounds (6.1) and (6.2) for $\gamma^\lambda(t)$ is similar.

We now wish to improve the bound (6.2) to

$$|\xi_\lambda(t)| \leq C \tag{6.5}$$

for $0 \leq t \leq T$. We begin by defining a function Q that depends on our co-ordinate systems. Let $\chi_1(\sigma), \ldots, \chi_N(\sigma)$ be a partition of unity with each χ_k supported in a single co-ordinate patch. Define $Q = \sum Q_k\chi_k$, where the local co-ordinate expression for Q_k is

$$Q_k(x, p) = \frac{1}{2}\langle p, G_\Sigma(x)^{-1}p\rangle + 1.$$

(We are abusing notation by using the same letter Q_k for the function on $T^*N\Sigma$ and its local co-ordinate expression.) Given (6.1) we may find a constant C such that

$$|\xi_\lambda(t)|^2 \leq CQ(\gamma_\lambda(t))$$

Thus bound (6.5) follows from an upper bound for Q along an orbit.

To establish such a bound we first estimate the time derivative of $Q_k(x_\lambda(t), p_\lambda(t))$. This derivative is given by the Poisson bracket.

$$\frac{d}{dt}Q_k(x_\lambda(t), p_\lambda(t)) = \{Q_k, L_\lambda\}(x_\lambda(t), p_\lambda(t), p_\lambda(t), r_\lambda(t))$$

Recall that the orthonormal frame $n_1(\sigma),\ldots,n_m(\sigma)$ giving our local co-ordinates consists of eigenvectors of $A(\sigma)$. Thus

$$L_\lambda = H_B + \lambda^2 H_O + E_\lambda$$

with

$$H_B(x,y,p,r) = Q_k(x,p) - \langle B(x,y)r, G_\Sigma(x)^{-1}p\rangle + \frac{1}{2}\langle B(x,y)r, G_\Sigma(x)^{-1}B(x,y)r\rangle + V(x,0),$$

$$H_O(x,y,p,r) = \frac{1}{2}\langle r,r\rangle + \frac{1}{2}\sum_i \omega_i^2 y_i^2$$

and

$$\begin{aligned} E_\lambda(x,y,p,r) = &\\ &\frac{1}{2}\left\langle \Big(p - B(x,y)r\Big), \Big((G_\Sigma(x) + C(x,\lambda^{-1}y))^{-1} - G_\Sigma(x)^{-1}\Big)\Big(p - B(x,y)r\Big)\right\rangle \\ &+ V(x,\lambda^{-1}y) - V(x,0) \end{aligned}$$

Since Q_k only depends on x and p any Poisson bracket $\{Q_k, F\}$ is given in local co-ordinates by

$$\{Q_k, F\} = \sum_i \frac{\partial Q_k}{\partial p_i}\frac{\partial F}{\partial x_i} - \frac{\partial Q_k}{\partial x_i}\frac{\partial F}{\partial p_i}.$$

Thus $\{Q_k, H_O\} = \{Q_k, Q_k\} = 0$. Using these formulas, together with (6.1) and (6.2) we find

$$\begin{aligned} \frac{d}{dt}Q_k(x_\lambda(t), p_\lambda(t)) &\le C\left(\|p_\lambda(t)\|^2 + \lambda^{-1}\|p_\lambda(t)\|^3\right) \\ &\le CQ_k(x_\lambda(t), p_\lambda(t)) \end{aligned} \tag{6.6}$$

Next, writing Hamilton's equations for $x_\lambda(t)$ and using (6.1) we find

$$\begin{aligned} |\dot{x}_\lambda(t)| &\le \left|\frac{\partial H_B}{\partial p}\right| \\ &\le CQ^{\frac{1}{2}}(x_\lambda(t), p_\lambda(t)) \end{aligned} \tag{6.7}$$

Since the cutoff functions, written in local co-ordinates, only depend on x_λ we find that

$$|\dot{\chi}_k| \le C|\dot{x}_\lambda| \le CQ^{\frac{1}{2}} \tag{6.8}$$

Now we show if we evaluate Q_k and Q_j at the same point $\gamma = (\sigma, n, \xi, \eta)$ with $|n|, |\eta| < C$ then

$$|Q_k(\gamma) - Q_j(\gamma)| \le CQ_k(\gamma)^{\frac{1}{2}}. \tag{6.9}$$

To see this, we first compute how our co-ordinates change. If $(\tilde{x}, \tilde{y}, \tilde{p}, \tilde{r})$ are the co-ordinates in the jth chart, obtained from the co-ordinates in the ith chart by a change of co-ordinates on Σ and a change of frame, then

$$\tilde{p} = Mp + b$$

$$\tilde{G}_\Sigma^{-1} = M^{-1}G_\Sigma^{-1}M^{-1}$$

where M is the $n \times n$ matrix with entries $\partial \tilde{x}_i / \partial x_j$ and b is a vector with components $\sum r_k y_l \partial \theta_{kl} / \partial x_i$ for an orthogonal matrix valued function $\theta(x)$ given by taking inner products of the elements of the old and new frames. Thus

$$\begin{aligned}
Q_j &= \langle \tilde{p}, \tilde{G}_\Sigma^{-1} \tilde{p} \rangle + 1 \\
&= Q_k + 2\langle b, M^{-1} G_\Sigma^{-1} p \rangle + \|b\|^2 + 1 \\
&\leq Q_k + C Q_k^{\frac{1}{2}}
\end{aligned}$$

This implies (6.9).

Now we are ready to establish a bound for Q along an orbit. Let $\dot{Q}$ denote $dQ(\gamma_\lambda(t))/dt$. Then

$$\begin{aligned}
\dot{Q} &= \sum_j \dot{Q}_j \chi_j + Q_j \dot{\chi}_j \\
&= \sum_j \dot{Q}_j \chi_j + \sum_{k,j} Q_j \dot{\chi}_j \chi_k
\end{aligned}$$

The first term is estimated using (6.6) yielding

$$\sum_j \dot{Q}_j \chi_j \leq C \sum_j Q_j \chi_j = CQ$$

To estimate the second term, note that since $\sum_k \chi_k = 1$, we have $\sum_k \dot{\chi}_k = 0$. Thus

$$\sum_{k,j} Q_k \dot{\chi}_j \chi_k = 0$$

so that

$$\begin{aligned}
\sum_{k,j} Q_j \dot{\chi}_j \chi_k &= \sum_{k,j} (Q_j - Q_k) \dot{\chi}_j \chi_k \\
&\leq CQ
\end{aligned}$$

by (6.8) and (6.9). Thus we have the differential inequality

$$\dot{Q} \leq CQ$$

which implies

$$Q(\gamma_\lambda(t)) \leq Q(\gamma_\lambda(0)) e^{Ct}$$

This implies (6.5)

Note that (6.7) implies

$$\|\dot{\sigma}_\lambda(t)\|, \|\dot{\sigma}^\lambda(t)\| < C \tag{6.10}$$

for $0 \leq t \leq T$.

We will now show that there exists $\epsilon > 0$ such that if

$$\lim_{\lambda \to \infty} \sup_{\tau \in [0,t]} \|\gamma_\lambda(\tau) - \gamma^\lambda(\tau)\| = 0 \tag{6.11}$$

holds for some $t = t_1 \leq T$ then (6.11) also holds for any $t \leq t_1 + \epsilon$. Since (6.11) holds for $t = 0$ by the assumption on the initial conditions, this will complete the proof.

So assume that (6.11) holds for $t = t_1 \leq T$. To compare the two orbits for nearby times, we want to ensure that they lie in the same co-ordinate patch. There exists an $\epsilon_1 > 0$ such that γ_λ and γ^λ will lie in the same co-ordinate chart if $\|\sigma_\lambda - \sigma^\lambda\| < \epsilon_1$.

Choose λ_0 so that $\lambda > \lambda_0$ implies

$$\sup_{\tau \in [0, t_1]} \|\gamma_\lambda(\tau) - \gamma^\lambda(\tau)\| < \epsilon_1/3$$

Now fix $\lambda > \lambda_0$. For $t > t_1$

$$\begin{aligned} \|\sigma_\lambda(t) - \sigma^\lambda(t)\| &\leq \|\sigma_\lambda(t) - \sigma_\lambda(t_1)\| + \|\sigma_\lambda(t_1) - \sigma^\lambda(t_1)\| + \|\sigma^\lambda(t_1) - \sigma^\lambda(t)\| \\ &\leq 2|t - t_1|C + \epsilon_1/3 \end{aligned}$$

where C is the constant from (6.10). Thus if we choose $\epsilon < \epsilon_1/3C$ then γ_λ and γ^λ will lie in the same co-ordinate chart for $t \in [t_1, t_1 + \epsilon]$. Notice that we do not rule out the the chart changes with λ.

We now write down the differential equation for γ_λ and γ^λ in this common co-ordinate chart. Let $z \in \mathbb{R}^{2(n+m)}$ denote co-ordinates for $T^*N\Sigma$, i.e.,

$$z = \begin{bmatrix} x \\ y \\ p \\ r \end{bmatrix}.$$

Denote by z_λ the co-ordinates of γ_λ and by z^λ the co-ordinates of γ^λ. For a Hamiltonian H, let X_H denote the corresponding Hamiltonian vector field given in local co-ordinates by

$$X_H(z) = \begin{bmatrix} \partial H/\partial x(z) \\ \partial H/\partial y(z) \\ -\partial H/\partial p(z) \\ -\partial H/\partial r(z) \end{bmatrix}$$

Then

$$\frac{d}{dt} z_\lambda(t) = X_{H_B}(z_\lambda(t)) + X_{\lambda^2 H_O}(z_\lambda(t)) + X_{E_\lambda}(z_\lambda(t)) \tag{6.12}$$

Since H_O is quadratic, the vector field $X_{\lambda^2 H_O}$ is linear, given by

$$X_{\lambda^2 H_O}(z) = \lambda^2 D z$$

for a matrix D that is similar to a real antisymmetric matrix. It follows that (6.12) can be written in integral form

$$z_\lambda(t) = e^{\lambda^2(t-t_1)D} z_\lambda(t_1) + e^{\lambda^2 t D} \int_{t_1}^{t} e^{-\lambda^2 \tau D} \Big(X_{H_B}(z_\lambda(\tau)) + X_{E_\lambda}(z_\lambda(\tau)) \Big) d\tau$$

We may write a similar equation for the co-ordinates of γ^λ and obtain

$$z_\lambda(t) - z^\lambda(t) = e^{\lambda^2(t-t_1)D}(z_\lambda(t_1) - z^\lambda(t_1)) \\ +e^{\lambda^2 tD}\int_{t_1}^t e^{-\lambda^2\tau D}\Big(X_{H_B}\left(z_\lambda(\tau)\right) - X_{H_B}\left(z^\lambda(\tau)\right) + X_{E_\lambda}\left(z_\lambda(\tau)\right)\Big)d\tau$$

The harmonic oscillator evolution $e^{\lambda^2 tD}$ is similar to a rotation and therefore uniformly bounded. Moreover we have the estimates

$$\left\|X_{H_B}\left(z_\lambda(\tau)\right) - X_{H_B}\left(z^\lambda(\tau)\right)\right\| \le C\left\|z_\lambda(\tau) - z^\lambda(\tau)\right\|$$

and

$$\|X_{E_\lambda}\left(z_\lambda(\tau)\right)\| \le C\lambda^{-1}$$

These follow from (6.1) and (6.5) which imply that the co-ordinates for the orbits stay in compact sets. Thus

$$\left\|z_\lambda(t) - z^\lambda(t)\right\| = C\left\|z_\lambda(t_1) - z^\lambda(t_1)\right\| + C|t-t_1|\sup_{\tau\in[t_1,t_1+\epsilon]}\left\|z_\lambda(\tau) - z^\lambda(\tau)\right\| + C|t-t_1|\lambda^{-1}$$

If we now also insist that

$$\epsilon < 1/(2C)$$

then we find that

$$\frac{1}{2}\sup_{\tau\in[t_1,t_1+\epsilon]}\left\|z_\lambda(\tau) - z^\lambda(\tau)\right\| \le C\left\|z_\lambda(t_1) - z^\lambda(t_1)\right\| + C\epsilon\lambda^{-1}$$

Since we have only finitely many co-ordinate charts, there is a constant C so that

$$C^{-1}\left\|z_\lambda(\tau) - z^\lambda(\tau)\right\| \le \|\gamma_\lambda(\tau) - \gamma^\lambda(\tau)\| \le C\left\|z_\lambda(\tau) - z^\lambda(\tau)\right\|$$

in any chart. Thus we conclude that

$$\sup_{\tau\in[t_1,t_1+\epsilon]}\|\gamma_\lambda(\tau) - \gamma^\lambda(\tau)\| \le C\left\|\gamma_\lambda(t_1) - \gamma^\lambda(t_1)\right\| + C\epsilon\lambda^{-1}$$

This implies that

$$\lim_{\lambda\to\infty}\sup_{\tau\in[t_1,t_1+\epsilon]}\|\gamma_\lambda(\tau) - \gamma^\lambda(\tau)\| = 0$$

and completes the proof.

□

Proof of Theorem 3.2: We will show that there exists $\epsilon > 0$ such that if (3.5) holds for some $t = t_1 \le T$, then (3.5) also holds for any $t \le t_1 + \epsilon$. So assume that (3.5) holds for some $t = t_1 \le T$.

Define

$$\psi_\lambda(t) = \phi_{-t}^{\lambda^2 H_O}\circ\phi_t^{H_B+\lambda^2 H_O}(\gamma_0)$$

Choosing our co-ordinate charts as in the proof of Theorem 3.1, we find that for small enough ϵ, $\psi_\lambda(t)$ will stay in a single chart for $t \in [t_1, t_1 + \epsilon]$. This follows from the estimate (6.10) for $\gamma^\lambda(t) = \phi_t^{H_B+\lambda^2 H_O}(\gamma_0)$ and the fact that the harmonic oscillator motion $\phi_{-t}^{\lambda^2 H_O}$ keeps the base point σ fixed.

Let $w_\lambda(t)$ denote the local co-ordinates of $\psi_\lambda(t)$. In local co-ordinates, the evolution $\phi_{-t}^{\lambda^2 H_O}$ is given by multiplication by $e^{-t\lambda^2 D}$, and so

$$w_\lambda(t) = e^{-t\lambda^2 D} z^\lambda(t),$$

where D is the same matrix, similar to a real antisymmetric matrix, that appeared in the proof of Theorem 3.1, and $z^\lambda(t)$ are the co-ordinates of $\gamma^\lambda(t)$. Differentiating, we obtain

$$\frac{dw_\lambda(t)}{dt} = e^{-t\lambda^2 D} X_{H_B}(e^{t\lambda^2 D} w_\lambda(t)),$$

so that for $t \in [t_1, t_1 + \epsilon]$,

$$w_\lambda(t) = w_\lambda(t_1) + \int_{t_1}^{t} e^{-s\lambda^2 D} X_{H_B}(e^{s\lambda^2 D} w_\lambda(s)) ds \tag{6.13}$$

Now consider the family of $\mathbb{R}^{2(n+m)}$ valued functions on $[t_1, t_1+\epsilon]$ given by $\mathcal{W} = \{w_\lambda(\cdot) : \lambda > 0\}$. We will show for any sequence $\lambda_j \to \infty$, there is a subsequence $\lambda_{1,j}$ such that $w_{\lambda_{1,j}}$ converges uniformly to the same limit w_∞. This will imply that $w_\lambda \to w_\infty$ uniformly.

The estimates (6.1) and (6.5) of Theorem 3.1 and the fact that the matrices e^{-tD} are bounded uniformly in t imply that $\mathcal{W}$ is a bounded family. Moreover, from (6.13) and the boundedness of the orbits, it follows that

$$\|w_\lambda(t) - w_\lambda(t')\| \le C|t - t'|$$

so that $\mathcal{W}$ is equicontinuous. Suppose we are given a sequence $\lambda_j \to \infty$. Then, by Ascoli's theorem, there exists subsequence $\lambda_{1,j}$ so that $w_{\lambda_{1,j}}$ converges uniformly to w_∞. We wish to show that w_∞ is always the same, no matter which sequence we start with. Our assumption on t_1 implies that $w_{\lambda_{1,j}}(t_1)$ always converges to the same w_0, namely to the co-ordinates of $\phi_{t_1}^{\overline{H}_B}(\gamma_0)$. We will show that $w_\infty(t)$ is the orbit generated by the Hamiltonian $\overline{H}_B$ with initial condition w_0 at $t = t_1$.

Using the uniform boundedness of the matrices e^{-tD} in (6.13) we find that

$$w_\infty(t) = w_0 + \int_{t_1}^{t} e^{-s\lambda_{1,j}^2 D} X_{H_B}(e^{s\lambda_{1,j}^2 D} w_\infty(s)) ds + o(1)$$

as $j \to \infty$. Now $e^{s\lambda_{1,j}^2 D}$ is a symplectic map, being the Hamiltonian flow $\phi_{s\lambda_{1,j}^2}^{H_0}$ in local co-ordinates. It follows that

$$e^{-s\lambda_{1,j}^2 D} X_{H_B}(e^{s\lambda_{1,j}^2 D} w_\infty(s)) = X_{H_B \circ \phi_{s\lambda_{1,j}^2}^{H_0}}(w_\infty(s))$$

If we use the Fourier expansion

$$H_B \circ \phi_{s\lambda_{1,j}^2}^{H_0} = \sum_{\nu \in \mathbb{Z}^{m_0}} e^{is\lambda_{1,j}^2 \langle \nu, \omega \rangle} F_\nu$$

we find that

$$X_{H_B\circ\phi^{H_0}_{s\lambda^2_{1,j}}} = \sum_{\nu\in\mathbb{Z}^{m_0}} e^{is\lambda^2_{1,j}\langle\nu,\omega\rangle} X_{F_\nu}$$

so that

$$w_\infty(t) = w_0 + \sum_{\nu\in\mathbb{Z}^{m_0}} \int_{t_1}^t e^{is\lambda^2_{1,j}\langle\nu,\omega\rangle} X_{F_\nu}(w_\infty(s))ds + o(1)$$

Taking j to infinity and using the Riemann-Lebesgue lemma, we find that

$$\begin{aligned} w_\infty(t) &= w_0 + \sum_{\nu\in\mathbb{Z}^{m_0}:\langle\nu,\omega\rangle=0} \int_{t_1}^t X_{F_\nu}(w_\infty(s))ds \\ &= w_0 + \int_{t_1}^t X_{\overline{H}_B}(w_\infty(s))ds \end{aligned}$$

This identifies $w_\infty(t)$ as the orbit generated by $\overline{H}_B$ with initial condition w_0 at t_1, as claimed.

Now we have

$$\sup_{t\in[t_1,t_1+\epsilon]} \left\| e^{-t\lambda^2 D} z^\lambda(t) - w_\infty(t) \right\| \to 0$$

as $\lambda\to\infty$ which implies

$$\sup_{t\in[t_1,t_1+\epsilon]} \left\| z^\lambda(t) - e^{t\lambda^2 D} w_\infty(t) \right\| \to 0$$

This implies

$$\sup_{t\in[t_1,t_1+\epsilon]} \left\| \phi_t^{H_B+\lambda^2 H_0}(\gamma_0) - \phi_t^{\lambda^2 H_0}\circ\phi_t^{\overline{H}_B}(\gamma_0) \right\| \to 0$$

and completes the proof. $\square$

7. More co-ordinate expressions

In this section we give the co-ordinate expressions that will be needed in our proofs of the quantum theorems.

We begin by defining the second fundamental form, the Weingarten maps and the mean and scalar curvatures. Let X and Y be two vector fields tangent to Σ. Since the Lie bracket $[X,Y] = dY[X] - dX[Y]$ is tangent to Σ we find that

$$I\!I(X,Y) = P^N dX[Y] = P^N dY[X] + P^N[X,Y] = P^N dY[X]$$

is symmetric in X and Y. Here P^N denotes the projection onto the normal space. By definition, $I\!I(X,Y)$ is the second fundamental form. Given an orthonormal frame $n_1(\sigma),\ldots,n_m(\sigma)$ for the normal bundle, we have

$$I\!I(X,Y) = \sum_k \langle X, S_k Y\rangle n_k$$

for a collection of symmetric linear transformations S_k on the tangent space. These are called the Weingarten maps. Clearly $\langle X, S_k Y\rangle = \langle n_k, dX[Y]\rangle$. But, by differentiating $\langle n_k, X\rangle = 0$, we obtain $\langle dn_k[Y], X\rangle + \langle n_k, dX[Y]\rangle = 0$, so that the Weingarten maps can also be written as $S_k = -P^T dn_k$. Here P^T denotes the orthogonal projection onto the tangent space.

The mean curvature vector is given by

$$h = \frac{1}{n}\sum_{k=1}^{m} \operatorname{tr}(S_k) n_k \tag{7.1}$$

while the scalar curvature is

$$s = \frac{1}{n(n-1)}\sum_{k=1}^{m}((\operatorname{tr}(S_k))^2 - \operatorname{tr}(S_k^2)) \tag{7.2}$$

Recall that the local expression $G(x, y)$ for the pulled back metric on $N\Sigma$ has the block form (5.2). Initially, $G(x, y)$ is only defined for $\|y\| < \delta$. In our theorem, we wish to extend this metric to a complete Riemannian metric on all of $N\Sigma$. One way to achieve this is to join the induced metric for small $|y|$ to the metric $\langle\cdot,\cdot\rangle_1$ given by (3.3) for large $|y|$. Since the matrix for the metric $\langle\cdot,\cdot\rangle_1$ is

$$\begin{bmatrix} I & B \\ 0 & I \end{bmatrix}\begin{bmatrix} G_\Sigma & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} I & B \\ 0 & I \end{bmatrix}^T$$

the resulting metric on all of $N\Sigma$ would have the matrix

$$G(x, y) = \begin{bmatrix} I & B \\ 0 & I \end{bmatrix}\begin{bmatrix} G_\Sigma + \chi C & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} I & B \\ 0 & I \end{bmatrix}^T$$

where $\chi = \chi(|y|)$ is a cutoff function that equals 1 for $|y| < \epsilon$ and 0 for $|y| > \delta$. With this special form of the extended metric the local co-ordinate expression below remain true on all of $N\Sigma$ if C is replaced by χC. However, this special form of the extension is not required for our theorems.

Let $g(x, y) = \det(G(x, y)) = \det(G_\Sigma + C)$. Define

$$D_x = \begin{bmatrix} D_{x_1} \\ \vdots \\ D_{x_n} \end{bmatrix}, \quad D_y = \begin{bmatrix} D_{y_1} \\ \vdots \\ D_{y_m} \end{bmatrix}$$

The local co-ordinate expression for the operator $H_\lambda = -\frac{1}{2}\Delta + V(\sigma, n) + \lambda^4 W(\sigma, n)$ in the region $|y| < \delta$ is

$$\begin{aligned} H_\lambda &= -\frac{1}{2}g^{-1/2}\begin{bmatrix} D_x - BD_y \\ D_y \end{bmatrix}^T g^{1/2}\begin{bmatrix} (G_\Sigma + C)^{-1} & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} D_x - BD_y \\ D_y \end{bmatrix} + V(x, y) + \frac{\lambda^4}{2}\langle y, A(x)y\rangle \\ &= -\frac{1}{2}g^{-1/2}\Big((D_x - BD_y)^T g^{1/2}(G_\Sigma + C)^{-1}(D_x - BD_y) + D_y^T g^{1/2} D_y\Big) \\ &\qquad + V(x, y) + \frac{\lambda^4}{2}\langle y, A(x)y\rangle \end{aligned}$$

Local expressions for the densities on $N\Sigma$ are

$$\begin{aligned} d\mathrm{vol} &= \sqrt{g(x,y)}|d^n x||d^m y| \\ d\mathrm{vol}_\lambda &= \sqrt{g(x,y/\lambda)}|d^n x||d^m y| \\ d\mathrm{vol}_{N\Sigma} &= \sqrt{g(x,0)}|d^n x||d^m y| = \sqrt{g_\Sigma(x)}|d^n x||d^m y| \end{aligned}$$

where $g_\Sigma(x) = \det(G_\Sigma(x))$. Thus the multiplication operator M_λ appearing in (4.2) is multiplication by $f_\lambda^{-1/4}$ where

$$f_\lambda(x,y) = \frac{g(x,y/\lambda)}{g_\Sigma(x)}.$$

We may now compute the local expression for L_λ. Conjugation by D_λ results in every multiplication by a (possibly matrix valued) function $F(x,y)$ being replaced by multiplication by $F(x,y/\lambda)$, and every D_y being replaced by λD_y. Conjugation by M_λ simply puts a multiplication by $f_\lambda^{-1/4}$ to the right of the operator, and a multiplication by $f_\lambda^{1/4}$ to the left. In a co-ordinate system for a domain in $N\Sigma$ of the form $\{(\sigma,n) : \sigma \in \mathcal{U}, n \in N\Sigma_\sigma\}$ let $D = \begin{bmatrix} D_x \\ D_y \end{bmatrix}$ and $G_\lambda(x,y)$ be the scaled and extended metric taking into account the scaling of D_y as well as y. In other words

$$G_\lambda(x,y) = \begin{bmatrix} I & 0 \\ 0 & \lambda I \end{bmatrix} G(x,y/\lambda) \begin{bmatrix} I & 0 \\ 0 & \lambda I \end{bmatrix}. \tag{7.3}$$

Then

$$\begin{aligned} L_\lambda &= -\frac{1}{2} f_\lambda^{1/4} g(x,y/\lambda)^{-1/2} D^T g(x,y/\lambda)^{1/2} G_\lambda^{-1} D f_\lambda^{-1/4} + V(x,y/\lambda) + \frac{\lambda^2}{2}\langle y, A(x)y\rangle \\ &= -\frac{1}{2} g_\Sigma^{-1/2} f_\lambda^{-1/4} D^T f_\lambda^{1/4} g_\Sigma^{1/2} G_\lambda^{-1} f_\lambda^{1/4} D f_\lambda^{-1/4} + V(x,y/\lambda) + \frac{\lambda^2}{2}\langle y, A(x)y\rangle \end{aligned} \tag{7.4}$$

Thus in the region where $\|y\| < \delta\lambda$ we may use the explicit form of the metric to obtain

$$\begin{aligned} L_\lambda = -\frac{1}{2} f_\lambda^{-1/4} g_\Sigma^{-1/2} \begin{bmatrix} D_x - BD_y \\ D_y \end{bmatrix}^T g_\Sigma^{1/2} f_\lambda^{1/2} \begin{bmatrix} (G_\Sigma + C_\lambda)^{-1} & 0 \\ 0 & \lambda^2 I \end{bmatrix} \begin{bmatrix} D_x - BD_y \\ D_y \end{bmatrix} f_\lambda^{-1/4} \\ + V(x,y/\lambda) + \frac{\lambda^2}{2}\langle y, A(x)y\rangle, \end{aligned} \tag{7.5}$$

where $C_\lambda(x,y) = C(x,y/\lambda)$. Note that formally putting $f_\lambda = 1$ above, and replacing C_λ by 0, we obtain for the first line of (7.5)

$$-\frac{1}{2} g_\Sigma^{-1/2} \begin{bmatrix} D_x \\ D_y \end{bmatrix}^T \begin{bmatrix} I & -B \\ 0 & I \end{bmatrix}^T g_\Sigma^{1/2} \begin{bmatrix} G_\Sigma^{-1} & 0 \\ 0 & \lambda^2 I \end{bmatrix} \begin{bmatrix} I & -B \\ 0 & I \end{bmatrix} \begin{bmatrix} D_x \\ D_y \end{bmatrix}$$

which is the Laplace-Beltrami operator for the metric which in local co-ordinates is

$$\begin{bmatrix} I & B \\ 0 & I \end{bmatrix} \begin{bmatrix} G_\Sigma & 0 \\ 0 & \lambda^{-2} I \end{bmatrix} \begin{bmatrix} I & B \\ 0 & I \end{bmatrix}^T.$$

This is easily seen to be the matrix for the metric (3.3). This explains part of the origin of the $H_B + \lambda^2 H_O$. A more complete analysis (to which we now turn) is necessary to understand the origin of the term $K(\sigma)$.

Before beginning this, note that the local expressions for H_B and H_O are given by

$$H_B = \frac{1}{2}(D_x - B(x,y)D_y)^* G_\Sigma^{-1}(D_x - B(x,y)D_y) + K(x) + V(x,0) \tag{7.6}$$

and

$$H_0 = \frac{1}{2}D_y^* D_y + \frac{1}{2}\langle y, A(x)y\rangle \tag{7.7}$$

Here D_x^* and D_y^* denote the formal adjoints with respect to $d\mathrm{vol}_{N\Sigma}$ given by $D_x^* = -g_\Sigma^{-1/2} D_x^T g_\Sigma^{1/2}$, $D_y^* = -g_\Sigma^{-1/2} D_y^T g_\Sigma^{1/2} = -D_y^T$ and $B^* = g_\Sigma^{-1/2} B^T g_\Sigma^{1/2} = B^T$.

We now wish to perform a large λ expansion of L_λ. To state the error estimates precisely, we introduce the notation E_k to denote a smooth function of x and y that vanishes to kth order at $y = 0$, evaluated at $(x, y/\lambda)$. Roughly speaking, E_k behaves like $(y/\lambda)^k$ for small y/λ. The effect of differentiating such an error term is given by

$$\begin{aligned} \frac{\partial E_k}{\partial x_i} &= E_k \\ \frac{\partial E_k}{\partial y_i} &= \begin{cases} \lambda^{-1}E_{k-1} & \text{if } k \geq 1 \\ \lambda^{-1}E_0 & \text{if } k = 0 \end{cases} \end{aligned}$$

In our theorems we will always assume that the eigenvalues ω_j^2 of $A(\sigma)$ are constant. If we choose the orthonormal frame in the definition of our co-ordinates to consist of eigenvectors of $A(\sigma)$ then $\langle n, A(\sigma)n\rangle = \sum_j \omega_j^2 y_j^2$. We will make this substitution without further comment below.

Lemma 7.1 *In the region where $\|y\| < \delta\lambda$, the local expression for L_λ can be written*

$$L_\lambda = H_B + \lambda^2 H_0 + (D_x - BD_y)^* E_1 (D_x - BD_y) + E_1.$$

Proof: In a co-ordinate system for a domain in $N\Sigma$ of the form $\{(\sigma, n) : \sigma \in \mathcal{U}, n \in N\Sigma_\sigma\}$ let $D = \begin{bmatrix} D_x \\ D_y \end{bmatrix}$ and $G_\lambda(x,y)$ be given by (7.3). Setting $k_\lambda = (1/4)\ln f_\lambda$, we may write (7.4) as

$$L_\lambda = \frac{1}{2}(D - \partial k_\lambda)^* G_\lambda^{-1}(D - \partial k_\lambda) + V(x, y/\lambda) + \frac{\lambda^2}{2}\sum_j \omega_j^2 y_j^2 \tag{7.8}$$

where $\partial k_\lambda = \begin{bmatrix} \partial_x k_\lambda \\ \partial_y k_\lambda \end{bmatrix}$, $\partial k_\lambda^* = (\partial k_\lambda)^T$, and $D^* = -g_\Sigma^{-1/2} D^T g_\Sigma^{1/2}$. We further expand (7.8) to obtain

$$L_\lambda = \frac{1}{2}D^* G_\lambda^{-1} D + \frac{1}{2}\partial k_\lambda^* G_\lambda^{-1} \partial k_\lambda + \frac{1}{2}\sum_{i,j} g_\Sigma^{-1/2}\partial_i \left(g_\Sigma^{1/2} \left(G_\lambda^{-1}\right)_{i,j} \partial_j k_\lambda\right) + V(x, y/\lambda) + \frac{\lambda^2}{2}\sum_j \omega_j^2 y_j^2 \tag{7.9}$$

If $\|y\| < \lambda\delta$ then

$$G_\lambda^{-1}(x,y) = \begin{bmatrix} I & -B(x,y) \\ 0 & I \end{bmatrix}^T \begin{bmatrix} (G_\Sigma(x) + C(x, y/\lambda))^{-1} & 0 \\ 0 & \lambda^2 I \end{bmatrix} \begin{bmatrix} I & -B(x,y) \\ 0 & I \end{bmatrix} \tag{7.10}$$

so that in this region we obtain

$$\begin{aligned}
L_\lambda &= \frac{1}{2}(D_x - BD_y)^* G_\Sigma(x)^{-1}(D_x - BD_y) + \frac{\lambda^2}{2} D_y^* D_y \\
&\quad + (D_x - BD_y)^* E_1 (D_x - BD_y) + E_1 + \frac{\lambda^2}{2}\sum_i \left(\partial_{y_i}^2 k_\lambda + (\partial_{y_i} k_\lambda)^2\right) \\
&\quad + V(x, y/\lambda) + \frac{\lambda^2}{2}\sum_j \omega_j^2 y_j^2 \\
&= H_B + \lambda^2 H_O + (D_x - BD_y)^* E_1 (D_x - BD_y) + E_1 \\
&\quad + \frac{\lambda^2}{2}\sum_i \left(\partial_{y_i}^2 k_\lambda + (\partial_{y_i} k_\lambda)^2\right) - K(x)
\end{aligned}$$

Here we used $(\partial_x - B\partial_y)E_k = E_k$ and $\partial k_\lambda = \begin{bmatrix} E_1 \\ \lambda^{-1}E_0 \end{bmatrix}$, so that $(\partial_x - B\partial_y)k_\lambda = E_1$.

The lemma will follow if we can show

$$\frac{\lambda^2}{2}\sum_i \left(\partial_{y_i}^2 k_\lambda + (\partial_{y_i} k_\lambda)^2\right) = K(x) + E_1 \tag{7.11}$$

This requires a more careful expansion of f_λ. The first step is to uncover the geometrical meaning of the term $G_\Sigma(x) + C(x, y)$ occurring in the expression (5.2) for the metric. Note that

$$\langle dn_k[\sigma_i], \sigma_j\rangle = -\langle S_k\sigma_i, \sigma_j\rangle = -\langle \sigma_i, S_k\sigma_j\rangle = \langle \sigma_i, dn_k[\sigma_j]\rangle$$

and that

$$M_k = G_\Sigma^{-1}[\langle \sigma_i, S_k\sigma_j\rangle]$$

is the matrix for the Weingarten map S_k in the basis $\sigma_1, \ldots, \sigma_n$. Let S be the symmetric operator defined by $\langle n, I\!I(X, Y)\rangle = \langle X, SY\rangle$. Then $S = \sum_k y_k S_k$, and the matrix for S in the basis $\sigma_1, \ldots, \sigma_n$ is

$$M = M(x, y) = \sum_k y_k M_k(x)$$

A short calculation shows

$$G_\Sigma + C = G_\Sigma (I - M)^2 \tag{7.12}$$

Given the block form (5.2) of G and (7.12), we obtain

$$\begin{aligned}
f_\lambda = g_\lambda / g_\Sigma &= \det(G(x, y/\lambda))/\det(G_\Sigma(x)) \\
&= \det(G_\Sigma(x)(I - \lambda^{-1}M(x,y))^2)/\det(G_\Sigma(x)) \\
&= \det(I - \lambda^{-1}M(x,y))^2.
\end{aligned}$$

Thus

$$\begin{aligned}
k_\lambda &= \frac{1}{2}\ln(f_\lambda^{1/2}) = \frac{1}{2}\ln\det(I - \lambda^{-1}M) \\
&= \frac{1}{2}\mathrm{tr}\ln(I - \lambda^{-1}M) \\
&= -\frac{1}{2}\lambda^{-1}\mathrm{tr}(M) - \frac{1}{4}\lambda^{-2}\mathrm{tr}(M^2) + E_3 \\
&= -\frac{1}{2}\lambda^{-1}\sum_k y_k \mathrm{tr}(S_k) - \frac{1}{4}\lambda^{-2}\sum_{k,l} y_k y_l \mathrm{tr}(S_k S_l) + E_3
\end{aligned}$$

This implies that

$$\partial_{y_i} k_\lambda = -\frac{1}{2}\lambda^{-1}\mathrm{tr}(S_i) + \lambda^{-2}E_1 + \lambda^{-1}E_2$$

and

$$(\partial_{y_i})^2 k_\lambda = -\frac{1}{2}\lambda^{-2}\mathrm{tr}(S_i^2) + \lambda^{-2}E_1.$$

Thus

$$\begin{aligned}\frac{\lambda^2}{2}\sum_i \left(\partial_{y_i}^2 k_\lambda + (\partial_{y_i} k_\lambda)^2\right) &= -\frac{1}{4}\mathrm{tr}(S_i^2) + \frac{1}{8}(\mathrm{tr}(S_i))^2 + E_1 \\ &= \frac{1}{4}\left((\mathrm{tr}(S_i))^2 - \mathrm{tr}(S_i^2)\right) - \frac{1}{8}(\mathrm{tr}(S_i))^2 + E_1 \\ &= \frac{n(n-1)}{4}s - \frac{n^2}{8}\|h\|^2 + E_1\end{aligned}$$

Thus proves (7.11) and completes the proof □

We conclude this section by discussing the expression for $\overline{H}_B$ in local co-ordinates. We may define local annihilation and creation operators, using the co-ordinates $y_{\alpha,j}$ defined in Section 5, as

$$\begin{aligned}a_{\alpha,j} &= \frac{1}{\sqrt{2\omega_\alpha}}(\omega_{\alpha,j}y_{\alpha,j} + D_{y_{\alpha,j}}) \\ a^*_{\alpha,j} &= \frac{1}{\sqrt{2\omega_\alpha}}(\omega_{\alpha,j}y_{\alpha,j} - D_{y_{\alpha,j}})\end{aligned}$$

Then we find

$$\begin{aligned}I_\alpha &= \sum_j \left(-\frac{1}{2\omega_\alpha}D^2_{y_{\alpha,j}} + \frac{\omega_\alpha}{2}y^2_{\alpha,j}\right) \\ &= \sum_j \left(a^*_{\alpha,j}a_{\alpha,j} + \frac{1}{2}\right)\end{aligned}$$

We may also write H_B in terms of the annihilation and creation operators. We begin with

$$(B(x,y)D_y)_i = \sum_{\alpha,j,\beta,k} b^i_{\alpha,j,\beta,k} D_{y_{\alpha,j}} y_{\beta,k}.$$

Notice that the order of $D_{y_{\alpha,j}}$ and $y_{\beta,k}$ is irrelevant here, since b^i is antisymmetric in (α,j) and (β,k). Then we can use

$$\begin{aligned}D_{y_{\alpha,j}} &= \sqrt{\frac{\omega_\alpha}{2}}\left(a_{\alpha,j} - a^*_{\alpha,j}\right) \\ y_{\beta,k} &= \sqrt{\frac{1}{2\omega_\alpha}}\left(a_{\beta,k} + a^*_{\beta,k}\right)\end{aligned}$$

and substitute the resulting expression in (7.6). The resulting formula expresses H_B as a finite sum of terms involving product of 0, 2 or 4 annihilation or creation operators. The identities

$$e^{itH_O}a_{\alpha,j}e^{-itH_O} = e^{-it\omega_\alpha}a_{\alpha,j} \qquad e^{itH_O}a^*_{\alpha,j}e^{-itH_O} = e^{it\omega_\alpha}a^*_{\alpha,j} \tag{7.13}$$

lead to a finite sum

$$e^{itH_O}H_Be^{-itH_O} = \sum_{\nu\in\mathbb{Z}^{m_0}} e^{it\langle\nu,\omega\rangle}F_\nu$$

that defines the differential operators F_ν.

Lemma 7.2 *For $\varphi \in C_0^\infty(N\Sigma)$, $e^{-itH_O}\varphi \in \mathcal{D}(H_B)$ and*

$$e^{itH_O}H_B e^{-itH_O}\varphi = \sum_{\nu\in\mathbb{Z}^{m_0}} e^{it\langle\nu,\omega\rangle}F_\nu\varphi$$

where the operators F_ν are defined by the sum above.

Proof: It suffices to prove this for $\varphi \in C_0^\infty$ supported in a single co-ordinate patch, since a general $\varphi \in C_0^\infty$ can be written as a sum of such functions. Introducing our usual local co-ordinates x and y, we find that e^{-itH_O} is simply a harmonic oscillator time evolution in the y variables. Hence $e^{-itH_O}\varphi$ is in Schwartz space. This implies that $e^{-itH_O}\varphi \in \mathcal{D}(H_B)$, and that the expansion of H_B into a sum of terms involving products of $a_{\alpha,j}$ and $a^*_{\alpha,j}$ is valid when applied to $e^{-itH_O}\varphi$. To complete the proof, it remains to show that the identities (7.13) hold when applied to a function φ in Schwartz space. This follows from

$$\begin{aligned}\frac{d}{dt}e^{itH_O}a_{\alpha,j}e^{-itH_O}\varphi &= ie^{itH_O}[H_O, a_{\alpha,j}]e^{-itH_O}\varphi \\ &= i\omega_\alpha e^{itH_O}[a^*_{\alpha,j}a_{\alpha,j}, a_{\alpha,j}]e^{-itH_O}\varphi \\ &= i\omega_\alpha e^{itH_O}[a^*_{\alpha,j}, a_{\alpha,j}]a_{\alpha,j}e^{-itH_O}\varphi \\ &= -i\omega_\alpha e^{itH_O}a_{\alpha,j}e^{-itH_O}\varphi.\end{aligned}$$

□

8. Proofs of theorems in quantum mechanics

We begin with some analysis that allows us to transfer our considerations from $\mathbb{R}^{n+m}$ to the normal bundle $N\Sigma$. Let

$$d(x,\Sigma) = \inf\{\|x-\sigma\| : \sigma \in \Sigma\}$$

denote the distance to Σ in $\mathbb{R}^{n+m}$ and let

$$\mathcal{U}_\delta = \{x \in \mathbb{R}^{n+m} : d(x,\Sigma) < \delta\}$$

be the tubular neighbourhood of Σ that is diffeomorphic to $N\Sigma_\delta$. The first proposition shows that the time evolution in $L^2(\mathbb{R}^{n+m})$ under H_λ is approximately the same for large λ as the time evolution in $L^2(\mathcal{U}_\delta)$ under the same Hamiltonian, except with Dirichlet boundary conditions.

Proposition 8.1 *Suppose that $W, V \in C^\infty(\mathbb{R}^{n+m})$ with $W \geq 0$ and V bounded below. Suppose $W(x) = 0$ if and only if $x \in \Sigma$ and that $W(x) \geq w_0 > 0$ for large x.*

Suppose $\lambda \geq 1$, $\psi \in L^2(\mathbb{R}^{n+m})$, $\|\psi\| = 1$ and $\|H_\lambda\psi\| \leq C_1\lambda^2$, where $H_\lambda = -\frac{1}{2}\Delta + V + \lambda^4 W$. Then, given $\epsilon > 0$ there exists C_2 such that for all $t \in \mathbb{R}$

$$\|F_{(d\geq\epsilon)}e^{-itH_\lambda}\psi\| \leq C_2\lambda^{-1}. \tag{8.1}$$

Here $F_{(\cdot)}$ denotes multiplication by the characteristic function supported on the region indicated in the parentheses.

Define H_λ^δ be the operator in $L^2(\mathcal{U}_\delta)$ given by H_λ with Dirichlet boundary conditions on ∂U_δ. Then for all $t \in [0,T]$ and $0 < \epsilon < \delta$

$$\|F_{(d\leq\epsilon)}e^{-itH_\lambda}\psi - e^{-itH_\lambda^\delta}F_{(d\leq\epsilon)}\psi\| \leq C_3\lambda^{-1/4} \tag{8.2}$$

Here C_2 depends only on C_1 and ϵ and C_3 depends only on C_1, T and ϵ.

Remark: The power $1/4$ in (8.2) is not optimal.

Proof: By the assumption on ψ and the Schwarz inequality

$$\langle \psi, H_\lambda \psi\rangle \leq C_1\lambda^2$$

Without loss we may assume that $V \geq 0$, so that

$$\begin{aligned} &\frac{1}{2}\|\nabla\psi\|^2 \leq C_1\lambda^2 \\ &\langle\psi, W\psi\rangle \leq C_1\lambda^{-2} \end{aligned} \tag{8.3}$$

It follows that

$$C(\epsilon)\langle\psi, F_{(d\geq\epsilon)}\psi\rangle \leq \langle\psi, F_{(d\geq\epsilon)}W\psi\rangle \leq C_1\lambda^{-2}$$

which proves (8.1), since $e^{-itH_\lambda}\psi$ satisfies the same hypotheses as ψ.

For $0 < \epsilon_1 \leq \alpha$ we will need the estimate

$$\|F_{(\epsilon_1\leq d\leq\alpha)}\nabla\psi\| \leq C_4\lambda^{\frac{1}{2}}, \tag{8.4}$$

where C_4 depends only on α, ϵ_1 and C_1. To prove this, choose a function $\chi \in C_0^\infty(\mathbb{R}^{n+m})$, $0 \leq \chi \leq 1$, which is 1 in a neighbourhood of $\{x : \epsilon_1 \leq d(x,\Sigma) \leq \alpha\}$ and vanishes in a neighbourhood of Σ. Then

$$\|F_{(\epsilon_1\leq d\leq\alpha)}\nabla\psi\| = \|F_{(\epsilon_1\leq d\leq\alpha)}\nabla(\chi\psi)\| \leq \|\nabla(\chi\psi)\|.$$

The Schwarz inequality and integration by parts gives

$$\|\nabla(\chi\psi)\| \leq \|\Delta(\chi\psi)\|^{\frac{1}{2}}\|\chi\psi\|^{\frac{1}{2}}$$

so that (8.4) follows from

$$\|\Delta(\chi\psi)\| \leq C_5\lambda^2 \tag{8.5}$$

and (8.1). To prove (8.5) let $p = -i\nabla$ and calculate, as forms on $C_0^\infty \times C_0^\infty$

$$H_\lambda^2 = \frac{1}{4}|p|^4 + (V+\lambda^4W)^2 + \sum_j p_j(V+\lambda^4W)p_j - \frac{1}{2}(\Delta V + \lambda^4\Delta W) \tag{8.6}$$

It follows from (8.6) and the fact that C_0^∞ is a core for H_λ that $\chi\psi \in \mathcal{D}(H_\lambda)$ and

$$\|\frac{1}{2}p^2\chi\psi\|^2 \leq \|H_\lambda(\chi\psi)\|^2 + C\lambda^4,$$

or

$$\frac{1}{2}\|p^2\chi\psi\| \leq \sqrt{C}\lambda^2 + \|H_\lambda\psi\| + \|[\frac{1}{2}p^2, \chi]\psi\|.$$

The last term can be bounded by (8.3), yielding (8.5).

Let $\tilde{\chi}$ be a smooth function which satisfies $0 \leq \tilde{\chi} \leq F_{(d<\epsilon/2)}$ and $\tilde{\chi} = 1$ in a neighbourhood of Σ. Because of (8.1) (which holds at $t = 0$) it is enough to show

$$\|e^{itH_\lambda^\delta}\tilde{\chi}e^{-itH_\lambda}\psi - \tilde{\chi}\psi\| \leq C\lambda^{-1/4}$$

for $t \in [0, T]$. Let

$$\phi_{t,\lambda} = e^{itH_\lambda^\delta}\tilde{\chi}e^{-itH_\lambda}\psi - \tilde{\chi}\psi.$$

Integrating the derivative, we obtain

$$\begin{aligned}\phi_{t,\lambda} &= i\int_0^t e^{isH_\lambda^\delta}(H_\lambda^\delta\tilde{\chi} - \tilde{\chi}H_\lambda)e^{-isH_\lambda}\psi ds \\ &= \int_0^t e^{isH_\lambda^\delta}(\nabla\tilde{\chi}\cdot p - (i/2)\Delta\tilde{\chi})e^{-isH_\lambda}\psi ds,\end{aligned}$$

and thus

$$\|\phi_{t,\lambda}\|^2 = \int_0^t \langle e^{-isH_\lambda^\delta}\phi_{t,\lambda}, (\nabla\tilde{\chi}\cdot p - (i/2)\Delta\tilde{\chi})e^{-isH_\lambda}\psi\rangle ds.$$

Let $\tilde{\tilde{\chi}} = 1$ on the support of $\nabla\tilde{\chi}$ and $\tilde{\tilde{\chi}} = 0$ in a neighbourhood of Σ. Then from (8.4)

$$\begin{aligned}\|\phi_{t,\lambda}\|^2 &\leq \int_0^t \|\tilde{\tilde{\chi}}e^{-isH_\lambda^\delta}\phi_{t,\lambda}\|(\|\nabla\tilde{\chi}\cdot pe^{-isH_\lambda}\psi\| + C)ds \\ &\leq C\lambda^{\frac{1}{2}}\int_0^t \|\tilde{\tilde{\chi}}e^{-isH_\lambda^\delta}\phi_{t,\lambda}\|ds\end{aligned}$$

Now

$$\begin{aligned}\langle\phi_{t,\lambda}, H_\lambda^\delta\phi_{t,\lambda}\rangle &\leq 2\langle\tilde{\chi}e^{-itH_\lambda}\psi, H_\lambda^\delta\tilde{\chi}e^{-itH_\lambda}\psi\rangle + 2\langle\tilde{\chi}\psi, H_\lambda^\delta\tilde{\chi}\psi\rangle \\ &= \langle e^{-itH_\lambda}\psi, (H_\lambda\tilde{\chi}^2 + \tilde{\chi}^2H_\lambda + (\nabla\tilde{\chi})^2)e^{-itH_\lambda}\psi\rangle + \langle\psi, (H_\lambda\tilde{\chi}^2 + \tilde{\chi}^2H_\lambda + (\nabla\tilde{\chi})^2)\psi\rangle \\ &\leq C\lambda^2,\end{aligned}$$

by the Schwarz inequality. Thus, following the proof of (8.1),

$$\|\tilde{\tilde{\chi}}e^{-isH_\lambda^\delta}\phi_{t,\lambda}\| \leq C\lambda^{-1}$$

so that

$$\|\phi_{t,\lambda}\|^2 \leq C\lambda^{\frac{1}{2}}\lambda^{-1}$$

which gives (8.2). □

Since the subset $\mathcal{U}_\delta \subset \mathbb{R}^{n+m}$ is diffeomorphic to $N\Sigma_\delta \subset N\Sigma$, we may think of $H_\lambda^\delta = -\frac{1}{2}\Delta + V + \lambda^4 W$ as acting in $L^2(N\Sigma_\delta, d\mathrm{vol})$ with Dirichlet boundary conditions on $\partial N\Sigma_\delta$, where the volume form $d$vol and the Laplace operator Δ are computed using the pulled back metric, and V and W are now the pull backs of the corresponding functions on $\mathcal{U}_\delta$. We may now extend the metric, and the potentials V and W, from $N\Sigma_\delta$ to all of $N\Sigma$, as explained in Section 4 above. Recall that the extended metric is assumed to be complete, that the extended V is bounded and that $W = \langle n, A(\sigma)n\rangle$ on all of $N\Sigma$. We thus obtain an operator H_λ acting in $L^2(N\Sigma, d\mathrm{vol})$. Since the extended metric is complete, H_λ is essentially self-adjoint on C_0^∞. Then it makes sense to talk about e^{-itH_λ}.

A proposition analogous to Proposition 8.1 holds in this situation, allowing us to approximate the evolution under H_λ^δ with an evolution under H_λ. For the purposes of this proposition, it does not matter how the extensions are made, as long as the conditions on the potentials hold, and the state ψ that we use for the comparison satisfies $\|H_\lambda\psi\| \leq C\lambda^2$. Since the statement and proof of this proposition are nearly identical to Proposition 8.1 we omit them.

Having justified the transfer of our considerations to $L^2(N\Sigma, d\mathrm{vol}_{N\Sigma})$, we now turn to the proof of Theorem 4.1.

Before beginning, we need some quantum energy bounds.

Lemma 8.2 *Let L_λ be as in Theorem 4.1 and $L_{0,\lambda} = H_B + \lambda^2 H_O$. Let $L_{\sharp\lambda}$ denote either of these operators and $R_{\sharp\lambda} = \left(\lambda^{-2}L_{\sharp\lambda} + 1\right)^{-1}$. Let $F_2 = F_{(|n|/\lambda<\epsilon)}$ be a smooth cutoff to the indicated region. When $\epsilon < \delta$, this cutoff function is supported in the region of $N\Sigma$ where the metric is explicitly defined. Let $\chi(\sigma)$ be a cutoff with support in a single co-ordinate patch. Then, for small enough ϵ and large λ,*

$$\|\langle n\rangle R_{\sharp\lambda}^{1/2}\| + \|\chi F_2 D_y R_{\sharp\lambda}^{1/2}\| + \|\lambda^{-1}\chi F_2 D_x R_{\sharp\lambda}^{1/2}\| \leq C \tag{8.7}$$

If l is a non-negative integer and α, β are multi-indices with $l + |\alpha| + |\beta| \leq 2$, then

$$\|\chi F_2 \langle n\rangle^l (\lambda^{-1}D_x)^\alpha D_y^\beta R_{\sharp\lambda}\| \leq C. \tag{8.8}$$

In addition, if l is a positive integer and $|\alpha| + |\beta| \leq 2$, then

$$\|\chi F_2 \langle n\rangle^l (\lambda^{-1}D_x)^\alpha D_y^\beta R_{\sharp\lambda}^{l+1}\| \leq C. \tag{8.9}$$

Here $\langle n\rangle = \sqrt{1+|n|^2}$.

Proof: Without loss of generality we can assume that $V \geq 1$. Set $f = \chi F_2$. Then $f \in C_0^\infty$ with $0 \leq f \leq 1$. Using (7.8) we see that

$$L_\lambda \geq \frac{1}{2}(D - \partial k_\lambda)^* f G_\lambda^{-1} f (D - \partial k_\lambda) + \frac{\lambda^2}{2}\sum_j \omega_j^2 y_j^2.$$

In the region where $f > 0$ we can use (7.9) to obtain

$$f \begin{bmatrix} I & -B \\ 0 & I \end{bmatrix}^T \begin{bmatrix} I & 0 \\ 0 & \lambda^2 I \end{bmatrix} \begin{bmatrix} I & -B \\ 0 & I \end{bmatrix} f \leq C f G_\lambda^{-1} f$$

Using $\lambda^{-2}R_\lambda^{1/2}(L_\lambda+\lambda^2)R_\lambda^{1/2}=1$ we obtain

$$\|fD_yR_\lambda^{1/2}\|\le C \tag{8.10}$$

$$\lambda^{-1}\|f(D_x-BD_y-\partial_xk_\lambda+B\partial_yk_\lambda)R_\lambda^{1/2}\|\le C \tag{8.11}$$

$$\|\langle n\rangle R_\lambda^{1/2}\|\le C. \tag{8.12}$$

On the support of f, $\partial_xk_\lambda-B\partial_yk_\lambda$ is bounded. Thus, using (8.10) and $\|B\|\le C|n|$ we obtain $\lambda^{-1}\|fD_xR_\lambda^{1/2}\|\le C$. This proves (8.7) for R_λ. The proof for $R_{0,\lambda}$ is similar.

Define U by $L_\lambda=\frac{1}{2}D^*G_\lambda^{-1}D+U$. Then, using (7.10) we calculate

$$\begin{aligned}L_\lambda f^2L_\lambda &= \frac{1}{4}(fD^*G_\lambda^{-1}D)^*(fD^*G_\lambda^{-1}D)+D^*G_\lambda^{-1}f^2UD+(Uf)^2\\ &+\frac{1}{2}D^*G_\lambda^{-1}[D,f^2U]+\frac{1}{2}[Uf^2,D^*]G_\lambda^{-1}D\end{aligned}$$

The last two terms above combine to give a multiplication operator given by a function which is easily shown to be bounded below by

$$-\tilde\chi^2\tilde F_2^2(1+\lambda^2|y|^2)$$

where $\tilde\chi$ and $\tilde F_2$ are like χ and F_2, with slightly expanded support. It follows that

$$\frac{\lambda^{-4}}{4}\|fD^*G_\lambda^{-1}DR_\lambda\|^2+\lambda^{-4}\|fG_\lambda^{-1/2}|U|^{1/2}DR_\lambda\|^2+\lambda^{-4}\|fUR_\lambda\|^2\le 1+\lambda^{-4}\|\tilde\chi\tilde F_2\langle n\rangle\lambda R_\lambda\|^2$$

The right side is bounded by (8.7). From $\lambda^{-2}\|fUR_\lambda\|\le C$ we obtain $\|f\langle n\rangle^2R_\lambda\|\le C$, which proves (8.8) when $l=2$. From

$$\lambda^{-2}\|fG_\lambda^{-1/2}|U|^{1/2}DR_\lambda\|\le C$$

we obtain

$$\lambda^{-1}\|f\langle n\rangle(D_x-BD_y)R_\lambda\|\le C$$

and

$$\|f\langle n\rangle D_yR_y\|\le C$$

which then gives

$$\|f\langle n\rangle\lambda^{-1}D_xR_\lambda\|\le C.$$

This proves (8.8) when $l=1$. Finally we consider the consequences of $\lambda^{-2}\|fD^*G_\lambda^{-1}DR_\lambda\|\le C$. This is equivalent to

$$\lambda^{-2}\|D^*G_\lambda^{-1}DfR_\lambda\|\le C$$

since the commutator term can be bounded using (8.7). We thus must examine the operator $D^*G_\lambda^{-1}D$ acting on functions of compact support in $\mathbb{R}^{n+m}$ contained in a domain of the form $\Theta_\lambda=\{(x,y):$

$|x| < r, |y| < \epsilon\lambda\}$ When we rescale $y \to \lambda y$ and $D_y \to \lambda^{-1} D_y$, the operator $D^* G_\lambda^{-1} D$ goes over to an elliptic operator E independent of λ operating on functions of compact support in a domain $\Theta = \{(x, y) : |x| < r, |y| < \epsilon\}$. The smooth coefficients of the operator E are bounded in Θ. It follows that if $|\alpha| + |\beta| \le 2$

$$\|D_x^\alpha D_y^\beta \psi\| \le C\|E\psi\|$$

for ψ with support in Θ. When we scale back again this implies

$$\lambda^{-2}\|D_x^\alpha (\lambda D_y)^\beta f R_\lambda\| \le C$$

or

$$\|(\lambda^{-1} D_x)^\alpha D_y^\beta f R_\lambda\| \le C.$$

Again, the commutator term which arises from moving f to the left can be bounded using (8.7). This takes care of the case $l = 0$ in (8.8). We have thus proved (8.8) for R_λ. The proof for $R_{0,\lambda}$ is similar.

We now turn to (8.9). We give the proof for R_λ. The proof for $R_{0,\lambda}$ is similar. We first show that

$$\|f\langle n\rangle^l R_\lambda^l\| \le C. \tag{8.13}$$

We write $f = f f_1^l$ where f_1 has slightly larger support than f and is of the form $h_1(x)h_2(|y|/\lambda)$. Writing $f_1\langle n\rangle = g$, we have

$$\begin{aligned} g^l R_\lambda^l &= g R_\lambda g^{l-1} R_\lambda^{l-1} + g[g^{l-1}, R_\lambda] R_\lambda^{l-1} \\ &= g R_\lambda g^{l-1} R_\lambda^{l-1} + g R_\lambda [\lambda^{-2} L_\lambda, g^{l-1}] R_\lambda^l \\ &= g R_\lambda g^{l-1} R_\lambda^{l-1} + g R_\lambda \big(D_y^* J_1 \langle n\rangle^{l-1} + \lambda^{-1}(D_x - B D_y)^* J_2 \langle n\rangle^{l-1} + J_3 \langle n\rangle^{l-1}\big) R_\lambda^l \end{aligned}$$

where J_1, J_2 and J_3 are bounded functions with support contained in $\mathrm{supp} f_1$. Thus, from (8.7)

$$\|g^l R_\lambda^l\| \le C\|g^{l-1} R_\lambda^{l-1}\| + C\|f_2 \langle n\rangle^{l-1} R_\lambda^{l-1}\|$$

where f_2 has slightly larger support than f_1. Thus (8.13) follows inductively.

We now let $A_{\alpha,\beta}$ denote $(\lambda^{-1} D_x)^\alpha D_y^\beta$ and take $A = A_{\alpha,\beta}$ with $|\alpha| + |\beta| \le 2$. Then

$$\|g^l A R_\lambda^{l+1}\| \le \|[A, g^l] R_\lambda^{l+1}\| + \|A f_2 g^l R_\lambda^{l+1}\|$$

where f_2 has slightly larger support than f_1. We have

$$\begin{aligned} \|A f_2 g^l R_\lambda^{l+1}\| &\le \|A f_2 R_\lambda g^l R_\lambda^l\| + \|A f_2 [g^l, R_\lambda] R_\lambda^l\| \\ &\le \|A f_2 R_\lambda\| \cdot \|g^l R_\lambda^l\| + \|A f_2 R_\lambda\| \cdot \|[g^l, \lambda^{-2} L_\lambda] R_\lambda^{l+1}\| \end{aligned}$$

and

$$[A, g_l] = \sum_{|\gamma|+|\mu|\le 1} g_{\gamma,\mu,l-1} (\lambda^{-1} D_x)^\gamma D_y^\mu$$

so that

$$\|[A, g_l]R_\lambda^{l+1}\| \le \sum_{|\gamma|+|\mu|\le 1} \|g_{\gamma,\mu,l-1}A_{\gamma,\mu}R_\lambda^l\|.$$

where $|g_{\gamma,\mu,l-1}| \le C(f_3\langle n\rangle)^{l-1}$ and where f_3 has slightly larger support than f_2. Similarly

$$[g^l, \lambda^{-2}L_\lambda] = \tilde{J}_1\langle n\rangle^{l-1}D_y + \tilde{J}_2\langle n\rangle^{l-1}(\lambda^{-1}D_x) + \tilde{J}_3\langle n\rangle^{l-1}$$

where $\tilde{J}_1$, $\tilde{J}_2$ and $\tilde{J}_3$ are bounded functions with support contained in $\mathrm{supp} f_1$. Thus

$$\|[g^l, \lambda^{-2}L_\lambda]R_\lambda^{l+1}\| \le \sum_{|\gamma|+|\mu|\le 1} \|\tilde{g}_{\gamma,\mu,l-1}A_{\gamma,\mu}R_\lambda^l\|$$

where $|\tilde{g}_{\gamma,\mu,l-1}| \le C(f_3\langle n\rangle)^{l-1}$. Thus again using induction, the result (8.9) follows. $\square$

Proof of Theorem 4.1: Since

$$\|e^{-itL_{0\lambda}}\psi - e^{-itL_\lambda}\psi\|^2 = 2\langle\psi,\psi\rangle - 2\,\mathrm{Re}\langle\psi, e^{itL_{0\lambda}}e^{-itL_\lambda}\psi\rangle$$

it suffices to show.

$$\lim_{\lambda\to\infty}\sup_{0\le t\le T}\left|\langle\psi, e^{itL_{0\lambda}}e^{-itL_\lambda}\psi\rangle - \langle\psi,\psi\rangle\right| = 0 \tag{8.14}$$

for a dense set of ψ in $L^2(N\Sigma, d\mathrm{vol}_{N\Sigma})$. Let $\psi \in C_0^\infty$. Our goal is to show (8.14).

As a first step, we insert an energy cutoff. Since $\|L_{\sharp\lambda}\psi\| \le C\lambda^2$ we have

$$\begin{aligned}\|F_{(L_{\sharp\lambda}/\lambda^2\ge\mu)}\psi\| &= \|F_{(L_{\sharp\lambda}/\lambda^2\ge\mu)}L_{\sharp\lambda}^{-1}\|\cdot\|L_{\sharp\lambda}\psi\|\\ &\le C\mu^{-1}\end{aligned}$$

Set

$$F_{\sharp 1} = F_{(L_{\sharp\lambda}/\lambda^2\le\mu)}$$

Then it suffices to show that for each fixed $\mu > 0$

$$\lim_{\lambda\to\infty}\sup_{0\le t\le T}\left|\langle F_{01}\psi, e^{itL_{0\lambda}}e^{-itL_\lambda}F_1\psi\rangle - \langle F_{01}\psi, F_1\psi\rangle\right| = 0. \tag{8.15}$$

We now need to show the quantum analogue of the fact in classical mechanics that the orbits stay in a bounded region of phase space if we watch the system for a time $T < \infty$ which is independent of λ. Using energy considerations it follows from Lemma 8.2 that $\langle n\rangle$ and D_y are bounded but only that D_x cannot grow faster than λ. We now seek a λ independent bound, showing that up to a fixed time T, not too much energy can be transferred from normal to tangential modes. In the quantum setting the statement

$$\|F_2D_x\chi e^{-itL_{\sharp\lambda}}F_{\sharp 1}\psi\| < C, \tag{8.16}$$

where F_2 is as in Lemma 8.2, will suffice.

We will prove this estimate when $L_{\sharp\lambda} = L_\lambda$, since the other case when $L_{\sharp\lambda} = L_{0\lambda}$is similar. Let $\{\chi_k^2(\sigma)\}$ be a partition of unity subordinate to a finite cover of co-ordinate charts. In other words, each χ_k^2 is supported in a single co-ordinate chart, and $\sum_k \chi_k^2 = 1$. We may assume that each χ_k is a smooth function only of σ. Define

$$Q = \sum_k \chi_k D_x^* G_\Sigma^{-1}(x) D_x \chi_k,$$

where, in each term, D_x and x are defined in terms of the co-ordinates for the chart in which χ_k is supported. We now want to cut Q off to the region where we have explicit expressions for the metric, and then add a constant to regain positivity. So let

$$\bar{Q} = F_2 Q F_2 + 1$$

Notice that Q and $\bar{Q}$ commute with F_2, since in local co-ordinates F_2 is a function of y alone. It is not difficult to show that both Q and $\bar{Q}$ are essentially self-adjoint on $C_0^\infty(N\Sigma)$. Define

$$q(t) = \langle F_1\psi, e^{itL_\lambda} \bar{Q} e^{-itL_\lambda} F_1 \psi\rangle.$$

Then (8.7) follows from

$$\sup\{q(t) : t \in [0,T]\} \leq C.$$

We will prove a differential inequality as in the classical case. We will need further estimates to bound the terms which arise when we compute $\dot{q}(t)$ and to prove an upper bound for $q(0)$.

Lemma 8.3 *Suppose F_1 is a smooth cutoff in the energy $\lambda^{-2}L_\lambda$. Then*

$$\left\| \left(\langle n\rangle^l (\lambda^{-1}D_x)^\alpha D_y^\beta \right) D_x^\gamma \chi_j F_2 F_1 \bar{Q}^{-1/2} \right\| \leq C$$

if $l + |\alpha| + |\beta| \leq 2$ and $|\gamma| = 1$.

Proof: We use the Helffer-Sjöstrand formula (see [D])

$$F_1 = \int g(z)(R_\lambda - z)^{-1} dz \wedge d\bar{z}$$

where we may take $g \in C_0^\infty(\mathbb{R}^2)$ with $|g(z)||\operatorname{Im} z|^{-N} \leq C_N$ for any N. (We are using the fact that $F_1(\lambda^{-2}L_\lambda) = \tilde{F}_1(R_\lambda)$ for $\tilde{F}_1 \in C_0^\infty(0,2)$. Let $A_1 = \langle n\rangle^\alpha (\lambda^{-1}D_x)^\beta D_y^\gamma \chi$ with $\chi \in C^\infty(\Sigma)$, supported in the jth co-ordinate patch, $\chi\chi_1 = \chi_1$, and let $F_{2,1}$ be a smooth function of $|n|/\lambda$ with $F_{2,1}F_2 = F_2$. Then

$$A_1 D_x^\gamma \chi_j F_2 F_1 \bar{Q}^{-1/2} = A_1 F_{2,1} F_1 D_x^\gamma \chi_j F_2 \bar{Q}^{-1/2} + A_1 F_{2,1} [D_x^\gamma \chi_j F_2, F_1] \bar{Q}^{-1/2}$$

Using (8.8), the first term is bounded by a constant times

$$\|A_1 F_{2,1} R_\lambda\| \cdot \|D_x^\gamma \chi_j F_2 \bar{Q}^{-1/2}\| \leq C$$

and it is thus sufficient to show

$$\|R_\lambda^{-1}[D_x^\gamma \chi_j F_2, F_1]\| \le C.$$

We compute from the Helffer-Sjöstrand formula

$$\|R_\lambda^{-1}[D_x^\gamma \chi_j F_2, F_1]\| \le C\|[D_x^\gamma \chi_j F_2, \lambda^{-2}L_\lambda]R_\lambda\| \tag{8.17}$$

For our present purposes we can write

$$L_\lambda = (D_x - BD_y)^* E_0 (D_x - BD_y) + \frac{\lambda^2}{2}(D_y^* D_y + \sum_j \omega^2 y_j^2) + E_0$$

and we thus obtain

$$\begin{aligned}[D_x^\gamma \chi_j F_2, \lambda^{-2}L_\lambda] &= \lambda^{-1} D_x^\gamma \chi_j (\nabla F_2 \cdot D_y + D_y \cdot \nabla F_2) \\ &\qquad + \lambda^{-2}[D_x^\gamma \chi_j, (D_x - BD_y)^* E_0 (D_x - BD_y)]F_2 + \lambda^{-2}E_0\end{aligned}$$

The first term gives a bounded contribution to (8.17) by Lemma 8.2. The second term can be written

$$\begin{aligned}&\Big(\lambda^{-1}(D_x - BD_y)^* E_0 \lambda^{-1}(D_x - BD_y) + D_y^* E_0 \lambda^{-1}(D_x - BD_y) \\ &+ \lambda^{-1}(D_x - BD_y)^* E_0 D_y + \lambda^{-2} E_0 (D_x - BD_y)\Big)\chi_j F_2 \\ &+ \lambda^{-1} D_x^\gamma \Big((\partial_x \chi_j)^T E_0 \lambda^{-1}(D_x - BD_y) + \lambda^{-1}(D_x - BD_y)^* E_0 \partial_x \chi_j\Big) F_2\end{aligned}$$

and again this gives a bounded contribution to (8.17) by Lemma 8.2.□

We now return to the proof of Theorem 4.1 and calculate

$$\dot{q}(t) = i\langle e^{-itL_\lambda}\psi, F_1[L_\lambda, \bar{Q}]F_1 e^{-itL_\lambda}\psi\rangle.$$

Let $F_{1,1}$ be a C_0^∞ function of $\lambda^{-2}L_\lambda$ with slightly larger support than F_1, so that $F_1 F_{1,1} = F_1$. We will show that

$$F_{1,1}[iL_\lambda, \bar{Q}]F_{1,1} \le C\bar{Q} \tag{8.18}$$

so that

$$q(t) \le e^{Ct} q(0).$$

First consider any term which arises when the cut-off $F_2 = F_{(|n|/\lambda<\epsilon)}$ is differentiated. The derivative F_2' has support in a region of the form $\{(\sigma, n) : \lambda\epsilon_1 < |n| < \lambda\epsilon_2\}$ so that $F_2'(\lambda/|n|)^l$ is bounded for any l. Thus $F_2' = \big(F_2'(\lambda/|n|)^l\big)\lambda^{-l}|n|^l$ so that according to Lemma 8.2, (8.9), such a term is bounded (and even decays faster that any inverse power of λ. Note that such a term occurring in the commutator $[L_\lambda, \bar{Q}]$ appears alongside $D_x^\alpha D_y^\beta$ with $|\alpha| + |\beta| \le 3$ but because we have an $F_{1,1}$ on the left and another on the right, (8.9) even allows $|\alpha| + |\beta| \le 4$ and we still obtain faster than any inverse power of λ decay.) Since $\bar{Q}$ contains the constant 1 such terms are harmless and we will ignore them. Thus we are left with showing

$$F_{1,1}F_2[iL_\lambda, Q]F_2F_{1,1} \leq C\bar{Q}. \tag{8.19}$$

We write

$$h_k = D_x^* G_\Sigma^{-1}(x) D_x$$

when the x refers to the kth co-ordinate patch. Then

$$\chi_k h_k \chi_k = \frac{1}{2}\left(\chi_k^2 h_k + h_k \chi_k^2\right) + (\partial_x \chi_k)^T G_\Sigma^{-1} \partial_x \chi_k$$

so that

$$\begin{aligned}[L_\lambda, Q] = \sum_k \Bigg(&\frac{1}{2}[L_\lambda, \chi_k^2]h_k + \frac{1}{2}h_k[L_\lambda, \chi_k^2] \\ &+ [L_\lambda, m_k] + \frac{1}{2}\chi_k^2[L_\lambda, h_k] + \frac{1}{2}[L_\lambda, h_k]\chi_k^2\Bigg)\end{aligned}$$

where $m_k = (\partial_x \chi_k)^T G_\Sigma^{-1} \partial_x \chi_k$. We must make use of some cancellation which occurs above so we write

$$\sum_k \frac{1}{2}[L_\lambda, \chi_k^2]h_k = \sum_{k,j} \frac{1}{2}[L_\lambda, \chi_k^2](h_k - h_j)\chi_j^2 + \sum_{k,j} \frac{1}{2}[L_\lambda, \chi_k^2]h_j \chi_j^2$$

and note that the second term on the right vanishes because $\sum_k \chi_k^2 = 1$. Thus we obtain

$$\begin{aligned}[L_\lambda, Q] = &\sum_{k,j} \frac{1}{2}[L_\lambda, \chi_k^2](h_k - h_j)\chi_j^2 + \frac{1}{2}\chi_j^2(h_k - h_j)[L_\lambda, \chi_k^2] \\ &+ [L_\lambda, \mathcal{M}] + \sum_k \frac{1}{2}\chi_k^2[L_\lambda, h_k] + \frac{1}{2}[L_\lambda, h_k]\chi_k^2\end{aligned}$$

where $\mathcal{M} = \sum_k m_k$.

In the term $[L_\lambda, \chi_k^2](h_k - h_j)\chi_j^2$ we refer all operators to the jth co-ordinate patch. Thus

$$h_k - h_j = \tilde{D}_x^* \tilde{G}_\Sigma^{-1} \tilde{D}_x - D_x^* G_\Sigma^{-1} D_x$$

where $\sim$ refers to the kth co-ordinate system. We obtain (schematically) $\tilde{D}_x = M^T D_x + \lambda E_1 D_y$ where $M \tilde{G}_\Sigma^{-1} M^T = G_\Sigma^{-1}$. Hence

$$h_k - h_j = (\lambda E_1 D_y + E_0)D_x + \lambda^2 E_2 D_y D_y + \lambda E_1 D_y + E_0.$$

After some calculation we find

$$\begin{aligned}\sum_{k,j} &\frac{1}{2}[L_\lambda, \chi_k^2](h_k - h_j)\chi_j^2 + \frac{1}{2}\chi_j^2(h_k - h_j)[L_\lambda, \chi_k^2] \\ &= \sum_j \chi_j D_x^*(\lambda E_1 D_y + E_0)D_x \chi_j + \chi_j D_x^*(\lambda^2 E_2 D_y D_y + \lambda E_1 D_y + E_0) \\ &\quad + \tilde{\chi}_j\left(D_y^* \lambda^3 E_3 D_y D_y + \lambda^2 E_2 D_y D_y + \lambda E_1 D_y + E_0\right)\end{aligned} \tag{8.20}$$

where $\tilde{\chi}_j \in C^\infty(\Sigma)$ with $\mathrm{supp}\tilde{\chi}_j$ contained in the jth co-ordinate patch. Noticing the presence of F_2 in (8.19) and using Lemma 8.3 with $\alpha = 0$ along with (8.9) of Lemma 8.2, we see that the terms in (8.20) give a contribution to the left side of (8.19) which is bounded by $C\bar{Q}$.

We can re-expand $\mathcal{M} = \mathcal{M}(\sigma)$ writing $\mathcal{M} = \sum_k \mathcal{M}\chi_k^2$ and then we find

$$[L_\lambda, \mathcal{M}] = \sum_k \chi_k (D_x^* E_0 + \lambda E_1 D_y + E_0)$$

which is readily handled by Lemma 8.3 and (8.9) of Lemma 8.2. We now expand the terms involving $[L_\lambda, h_k]$. After some calculation we obtain

$$\begin{aligned}
\sum_k \frac{1}{2}\chi_k^2[L-\lambda, h_k] + \frac{1}{2}[L_\lambda, h_k]\chi_k^2 \\
&= \sum_k \chi_k D_x^* \left(E_1 D_x + \lambda E_1 D_y + \lambda E_1 + E_0\right) D_x \chi_k \\
&\quad + \sum_k \chi_k D_x^* \left((\lambda^2 E_2 + \lambda E_1) D_y D_y + \lambda E_1 D_y + \lambda E_1 + E_0\right) \\
&\quad + \sum_k \chi_k \left((\lambda^2 E_2 + \lambda E_1) D_y D_y + (\lambda E_1 + E_0) D_y + \lambda E_1 + E_0 + \lambda^{-1} E_0\right) \\
&\quad + \sum_k \left(\chi_k D_x^* E_1 D_x + \tilde{\chi}_k E_q D_x\right)
\end{aligned}$$

where $\tilde{\chi}_k \in C^\infty(\Sigma)$ has support in the kth co-ordinate patch with $\tilde{\chi}_k \chi_k = \chi_k$. These terms are also easily handled with a combination of Lemma 8.2, (8.9) and Lemma 8.3. This completes the proof of (8.19) and shows

$$q(t) \le e^{Ct} q(0).$$

Finally

$$q(0) = \langle F_1\psi, \bar{Q} F_1 \psi\rangle$$

has λ dependence and must be bounded uniformly in λ. But this follows from Lemma 8.3 (with $l = \alpha = \beta = 0$) and the fact that $\|\bar{Q}^{1/2}\psi\|^2 = \langle \psi, \bar{Q}\psi\rangle < \infty$, independently of λ.

We now return to (8.15). We introduce a stronger cutoff in the n variable by restricting $|n|/\lambda^s < 1$ where $s \in (0,1)$. Thus let $F_3 = F_{(|n|/\lambda^s < 1)}$ be a smooth cutoff the the indicated region. We note that

$$\|(1-F_3)F_1\| \le \lambda^{-s}\|(1-F_1)\lambda^s/|n|\| \cdot \|\langle n\rangle F_1\| \le C\lambda^{-s}$$

by (8.7) of Lemma 8.2. Thus it is sufficient to prove

$$\lim_{\lambda\to\infty} \sup_{t\in[0,T]} \left|\langle F_{0,1}\psi, e^{itL_{0,\lambda}} F_3 e^{-itL_\lambda} F_1\psi\rangle - \langle F_{0,1}\psi, F_3 F_1 \psi\rangle\right| = 0$$

By the fundamental theorem of calculus we obtain

$$\begin{aligned}
\langle F_{0,1}\psi, e^{itL_{0,\lambda}} F_3 e^{-itL_\lambda} F_1\psi\rangle - \langle F_{0,1}\psi, F_3 F_1\psi\rangle \\
&= i\int_0^t \langle F_{0,1}\psi, e^{isL_{0,\lambda}} \left([L_{0,\lambda}, F_3] + F_3(L_{0,\lambda} - L_\lambda)\right) e^{-isL_\lambda} F_1\psi\rangle ds
\end{aligned} \tag{8.21}$$

The term $[L_{0,\lambda}, F_3]$ contains derivatives of F_3 and thus by Lemma 8.2, (8.9) its contribution to (8.21) decays faster than any inverse power of λ uniformly for $t \in [0,T]$. According to Lemma 7.1, on the support of F_3 we have

$$L_\lambda - L_{0,\lambda} = \sum_k \chi_k \Big((D_x - BD_y)^* E_1 (D_x - BD_y) + E_1 \Big) \chi_k.$$

Thus, aside from terms involving derivatives of F_3, which again can be handled by Lemma 8.2, (8.9) we need only show that

$$\lim_{\lambda\to\infty} \sup_{s\in[0,T]} \left| \langle F_{0,1} e^{isL_{0,\lambda}} \psi, \Big(F_2 \chi_k (D_x - BD_y)^* F_3 E_1 (D_x - BD_y) F_2 \chi_k + \chi_k^2 F_3 E_1 \Big) F_1 e^{-isL_\lambda} \psi \rangle \right| = 0$$

Now

$$\|F_3 E_1 F_1\| \le C\lambda^{-1} \|\langle n \rangle F_1\| \le C\lambda^{-1}$$

so we need only bound the product

$$\|(D_x - BD_y)\chi_k F_2 F_{0,1} e^{-sL_{0,\lambda}} \psi\| \cdot \|F_3 E_1\| \cdot \|(D_x - BD_y)\chi_k F_2 F_1 e^{-isL_\lambda} \psi\|.$$

By Lemma 8.2, (8.8)

$$\|BD_y \chi_k F_2 F_{\sharp,1}\| \le C$$

and by (8.16)

$$\sum_{s\in[0,T]} \|D_x \chi_k F_2 F_{\sharp,1} e^{isL_{\sharp\lambda}} \psi\| \le C.$$

Finally

$$\|F_3 E_1\| \le C\lambda^s/\lambda = C\lambda^{s-1},$$

which proves (8.15) and thus completes the proof of the theorem. $\square$

Proof of Theorem 4.2: To prove the theorem, it suffices to show that for any $\psi \in C_0^\infty(N\Sigma)$,

$$\lim_{\lambda\to\infty} \sup_{0\le t\le T} \left\| \left(e^{-it(H_B+\lambda^2 H_O)} - e^{-it\lambda^2 H_O} e^{-it\overline{H}_B} \right) \psi \right\|^2 = 0$$

This can be rewritten as

$$\lim_{\lambda\to\infty} \sup_{0\le t\le T} \left| \langle \psi_{t,\lambda} - e^{-it\overline{H}_B} \psi, e^{-it\overline{H}_B} \psi \rangle \right| = 0 \tag{8.22}$$

where

$$\psi_{t,\lambda} = e^{it\lambda^2 H_O} e^{-it(H_B+\lambda^2 H_O)} \psi$$

We will show that for any $\varphi \in L^2(N\Sigma, d\mathrm{vol}_{N\Sigma})$

$$\lim_{\lambda\to\infty} \sup_{0\le t\le T} \left| \langle \psi_{t,\lambda} - e^{-it\overline{H}_B} \psi, \varphi \rangle \right| = 0, \tag{8.23}$$

which will imply (8.22).

This implication follows from the general fact that if $\psi_{t,\lambda}$ converges to some $\psi_{t,\infty}$ with

$$\sup_{0\leq t\leq T} \|\psi_{t,\infty}\| \leq C$$

in the sense that

$$\lim_{\lambda\to\infty} \sup_{0\leq t\leq T} |\langle \psi_{t,\lambda} - \psi_{t,\infty}, \varphi\rangle| = 0$$

then, for any continuous function φ_t from $[0,T]$ into $L^2(N\Sigma, d\mathrm{vol}_{N\Sigma})$,

$$\lim_{\lambda\to\infty} \sup_{0\leq t\leq T} |\langle \psi_{t,\lambda} - \psi_{t,\infty}, \varphi_t\rangle| = 0.$$

To see this, pick an orthonormal basis $\{\varphi_n\}$. Then

$$\begin{aligned}
&\sup_{0\leq t\leq T} |\langle \psi_{t,\lambda} - \psi_{t,\infty}, \varphi_t\rangle| \\
&\qquad \leq \sup_{0\leq t\leq T} \left|\sum_{n=1}^{N} \langle \psi_{t,\lambda} - \psi_{t,\infty}, \varphi_n\rangle\langle \varphi_n, \varphi_t\rangle\right| + \sup_{0\leq t\leq T} \left|\sum_{n=N+1}^{\infty} \langle \psi_{t,\lambda} - \psi_{t,\infty}, \varphi_n\rangle\langle \varphi_n, \varphi_t\rangle\right| \\
&\qquad \leq C \sup_{0\leq t\leq T} \sum_{n=1}^{N} |\langle \psi_{t,\lambda} - \psi_{t,\infty}, \varphi_n\rangle| + C \sup_{0\leq t\leq T} \|(1-P_N)\varphi_t\|,
\end{aligned}$$

where P_N denotes the orthogonal projection onto the span of $\varphi_1, \ldots, \varphi_N$. The first term on the right tends to zero as $\lambda \to \infty$, by assumption. Hence

$$\limsup_{\lambda\to\infty} \sup_{0\leq t\leq T} |\langle \psi_{t,\lambda} - \psi_{t,\infty}, \varphi_t\rangle| \leq C \sup_{0\leq t\leq T} \|(1-P_N)\varphi_t\|$$

But $\{\phi_t : t \in [0,T]\}$ is compact and $1 - P_N$ goes to zero uniformly on compact sets. Therefore the right side tends to zero as $N \to \infty$.

Thus it suffices to prove (8.23), which we will do in two steps. First, we will show that for every sequence $\lambda_j \to \infty$, there exists a subsequence μ_j and a bounded, weakly continuous $L^2(N\Sigma, d\mathrm{vol}_{N\Sigma})$ valued function $\psi_{t,\infty}$ such that

$$\sup_{0\leq t\leq T} \left|\langle \psi_{t,\mu_j} - \psi_{t,\infty}, \varphi\rangle\right| \to 0 \tag{8.24}$$

for every $\varphi \in L^2(N\Sigma, d\mathrm{vol}_{N\Sigma})$. Then, to complete the proof, we will show that $\psi_{t,\infty}$ is always the same, and equal to $e^{-it\overline{H}_B}\psi$.

To take the first step, we begin with a sequence $\lambda_j \to \infty$. Let $\{\varphi_n\}$ be an orthonormal basis of vectors in $C_0^\infty(N\Sigma)$. Define

$$w_{n,\lambda}(t) = \langle \psi_{t,\lambda}, \varphi_n\rangle$$

Then for fixed n, $w_{n,\lambda}(t)$ are a family of functions of $t \in [0,T]$, uniformly bounded as $\lambda \to \infty$. Still for fixed n, this family is equicontinuous, since the derivative is bounded independently of λ. This follows from

$$\left|\frac{d}{dt}\langle\psi_{t,\lambda},\varphi_n\rangle\right| = \left|-i\langle e^{it\lambda^2 H_O}H_B e^{-it(H_B+\lambda^2 H_O)}\psi,\varphi_n\rangle\right|$$
$$= \left|\langle\psi_{t,\lambda},e^{it\lambda^2 H_O}H_B e^{-it\lambda^2 H_O}\varphi_n\rangle\right|$$
$$\leq \|\psi\|\cdot\|\sum_\nu e^{it\lambda^2\langle\nu,\omega\rangle}F_\nu\varphi_n\|$$
$$\leq \|\psi\|\cdot\sum_\nu\|F_\nu\varphi_n\| = C_n$$

The sum over ν is finite. Here we used (7.14), and that φ_n is in $C_0^\infty(N\Sigma)$, and therefore in the domain of F_ν.

Using Ascoli's theorem, we may now choose a subsequence λ_{j_1} of λ_j so that $w_{1,\lambda_{j_1}}(t)$ converges to some continuous function $w_{1,\infty}(t)$, uniformly in t for $t\in[0,T]$. Then we may choose a subsequence λ_{j_2} of λ_{j_1} with $w_{2,\lambda_{j_2}}(t)$ converging uniformly to some continuous function $w_{2,\infty}(t)$. Continuing in this way, and then taking a diagonal subsequence, we end up with a subsequence μ_j such that

$$\sup_{0\leq t\leq T}\left|w_{n,\mu_j}(t)-w_{n,\infty}(t)\right|\to 0$$

for every n. Notice that

$$\sum_{n=1}^N|w_{n,\infty}(t)|^2 = \lim_{j\to\infty}\sum_{n=1}^N|\langle\psi_{t,\mu_j},\varphi_n\rangle|^2$$
$$\leq \|\psi_{t,\mu_j}\|^2$$
$$= \|\psi\|^2$$

This implies that $\sum_{n=1}^\infty|w_{n,\infty}(t)|^2\leq\|\psi\|^2$, so that

$$\psi_{t,\infty} = \sum_n w_{n,\infty}(t)\varphi_n$$

is well defined with $\|\psi_{t,\infty}\|\leq\|\psi\|$. Clearly, for any n, $\langle\psi_{t,\mu_j}-\psi_{t,\infty},\varphi_n\rangle\to 0$ as $j\to\infty$. This implies (8.24)

Now take the second step of identifying $\psi_{t,\infty}$. Let $\varphi\in C_0^\infty(N\Sigma)$. Then

$$\begin{aligned}\langle\psi_{t,\mu_j},\varphi\rangle &= \langle\psi,\varphi\rangle + i\int_0^t\langle\psi_{s,\mu_j},e^{is\mu_j^2 H_O}H_B e^{-is\mu_j^2 H_O}\varphi\rangle ds\\ &= \langle\psi,\varphi\rangle + i\int_0^t\langle\psi_{s,\infty},e^{is\mu_j^2 H_O}H_B e^{-is\mu_j^2 H_O}\varphi\rangle ds\\ &\qquad + i\int_0^t\langle\psi_{s,\mu_j}-\psi_{s,\infty},e^{is\mu_j^2 H_O}H_B e^{-is\mu_j^2 H_O}\varphi\rangle ds\end{aligned}\tag{8.25}$$

Since $\varphi\in C_0^\infty(N\Sigma)$ the formula (7.14) implies that

$$|\langle\psi_{s,\mu_j}-\psi_{s,\infty},e^{is\mu_j^2 H_O}H_B e^{-is\mu_j^2 H_O}\varphi\rangle| \leq \sum_\nu|\langle\psi_{s,\mu_j}-\psi_{s,\infty},F_\nu\varphi\rangle|$$

Thus the second term of (8.25) tends to zero as $j \to \infty$. On the other hand

$$\begin{aligned}\lim_{j\to\infty}\int_0^t \langle\psi_{s,\infty}, e^{is\mu_j^2 H_O} H_B e^{-is\mu_j^2 H_O}\varphi\rangle ds &= \lim_{j\to\infty}\sum_\nu \int_0^t e^{is\mu_j^2\langle\nu,\omega\rangle}\langle\psi_{s,\infty}, F_\nu\varphi\rangle ds \\ &= \int_0^t \langle\psi_{s,\infty}, \overline{H}_B\varphi\rangle ds\end{aligned}$$

by the Riemann Lebesgue lemma. Thus, taking $j \to \infty$ in (8.25) we obtain

$$\langle\psi_{t,\infty}, \varphi\rangle = \langle\psi, \varphi\rangle + i\int_0^t \langle\psi_{s,\infty}, \overline{H}_B\varphi\rangle ds. \tag{8.26}$$

Now let $\tilde{\varphi}$ be in the domain of $\overline{H}_B$. Since C_0^∞ is a core for $\overline{H}_B$, we may use an approximation argument to replace φ with $e^{-is\overline{H}_B}\tilde{\varphi}$ and $\overline{H}_B\varphi$ with $\overline{H}_B e^{-is\overline{H}_B}\tilde{\varphi}$ in the equation above. We find, using (8.26),

$$\begin{aligned}\frac{d}{ds}\langle\psi_{s,\infty}, e^{-is\overline{H}_B}\tilde{\varphi}\rangle &= \frac{d}{dt}\langle\psi_{t,\infty}, e^{-is\overline{H}_B}\tilde{\varphi}\rangle\Big|_{t=s} + \frac{d}{dt}\langle\psi_{s,\infty}, e^{-it\overline{H}_B}\tilde{\varphi}\rangle\Big|_{t=s} \\ &= i\langle\psi_{s,\infty}, \overline{H}_B e^{-is\overline{H}_B}\tilde{\varphi}\rangle - i\langle\psi_{s,\infty}, \overline{H}_B e^{-is\overline{H}_B}\tilde{\varphi}\rangle \\ &= 0\end{aligned}$$

Thus $\langle\psi_{s,\infty}, e^{-is\overline{H}_B}\tilde{\varphi}\rangle$ is constant. But when $s = 0$, equation (8.26) implies $\langle\psi_{s,\infty}, e^{-is\overline{H}_B}\tilde{\varphi}\rangle = \langle\psi, \tilde{\varphi}\rangle$. Thus$\langle e^{is\overline{H}_B}\psi_{s,\infty} - \psi, \tilde{\varphi}\rangle = 0$ for every $\tilde{\varphi}$ in the domain of $\overline{H}_B$. This implies $e^{is\overline{H}_B}\psi_{s,\infty} = \psi$, or $\psi_{s,\infty} = e^{-is\overline{H}_B}\psi$, and completes the proof. □

9. References

[AD] L. Andersson and B.K. Driver, *Finite dimensional approximations to Wiener measure and path integral formulas on manifolds*, J. Funct. Anal. **165**, no. 2 (1999) 430–498.

[A1] V.I. Arnold, *Mathematical Methods of Classical Mechanics,* Springer-Verlag, 1978.

[A2] V.I. Arnold, V.V. Kozlov and A.I. Neishtadt, *Mathematical Aspects of Classical and Celestial Mechanics,* Encyclopædia of Mathematical sciences, Volume 3, Springer-Verlag, 1988.

[BS] Bornemann and Schütte, *Homogenization of Hamiltonian systems with a strong constraining potential,* Physica D, **102** (1997) 57-77

[C] Paul R. Chernoff, *Essential self-adjointness of powers of generators of hyperbolic equations*, J. Functional Analysis vol. 12, (1973) 401-414

[dC1] R.C.T. da Costa, *Quantum mechanics of a constrained particle,* Physical Review A, Volume 23 Number 4 (1981) 1982–1987.

[dC2] R.C.T. da Costa, *Constraints in quantum mechanics,* Physical Review A, Volume 25 Number 6 (1982) 2893–2900.

[D] E.B. Davies, *The functional calculus,* J. London Math. Soc. (2) **52** no. 1 (1995) 166–176.

[DE] P. Duclos and P. Exner, *Curvature-induced bound states in quantum waveguides in two and three dimensions,* Reviews in Mathematical Physics, **7**, no. 1 (1995) 73–102

[FK] A. Figotin and P. Kuchment, *Spectral properties of classical waves in high contrast media,* SIAM Journal of Applied Mathematics, **58** (1998) 683–702

[F] M. Friedlin, *Markov Processes and Differential Equations: Asymptotic Problems,* Lectures in Mathematics ETH, Birkhäuser, 1996.

[FW] M.I. Friedlin and A.D. Wentzell, *Random Perturbations of Dynamical Systems,* second edition, Springer, 1998.

[FH] R. Froese and I. Herbst, *Realizing holonomic constraints in classical and quantum mechanics,* Proceedings of the 1999 VAB–GIT international conference on differential equations.

[G] G. Gallavotti, *The Elements of Mechanics,* Springer-Verlag, 1983.

[GS] V. Guillemin and S. Sternberg, *Symplectic techniques in physics,* Cambridge University Press, 1984.

[HS1] B. Helffer and J. Sjöstrand, *Puits multiples en méchanique semi-classique V, Étude des mini-puits,* Current topics in partial differential equations, Kinokuniya Company Ltd., 133–186 (Volume in honour of S. Mizohata)

[HS2] B. Helffer and J. Sjöstrand, *Puits multiples en méchanique semi-classique VI, Cas des puits sous-variétés,* Annales de l'Institut Henri Poincaré, Physique Théorique, **46** no. 4 (1987) 353–372

[KZ] P. Kuchment and H. Zeng, *Convergence of spectra of mesoscopic systems collapsing onto a graph,* preprint (mp_arc preprint no. 00-308 at `http://www.ma.utexas.edu/mp_arc/`)

[M] K.A. Mitchell, *Geometric phase, curvature, and extrapotentials in constrained quantum systems* preprint at `http://xxx.lanl.gov/abs/quant-ph/0001059` 18 Jan 2000

[RU] H. Rubin and P. Ungar, *Motion under a strong constraining force,* Communications on pure and applied mathematics, **10** (1957) 28–42

[S] M. Schatzman, *On the eigenvalues of the Laplace operator on a thin set with Neumann boundary conditions,* Applicable Analysis, **61** (1996) 293–306

[Ta] F. Takens, *Motion under the influence of a strong constraining force,* in Global theory of dynamical systems, Springer lecture notes in mathematics, **819** (1980) 425–445

[To] J. Tolar, *On a quantum mechanical d'Alembert principle,* Group theoretical methods in physics, Springer lecture notes in physics, **313** (1988) 268–274