Upper bounds for the resonance counting function

of Schrödinger operators in odd dimensions

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Abstract

The purpose of this note is to provide a simple proof of the sharp polynomial upper bound for the resonance counting function of a Schrödinger operator in odd dimensions. At the same time we generalize the result to the class of super-exponentially decreasing potentials.

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Introduction

Let $H = -\Delta + V$ be a Schrödinger operator acting in $L^2(\mathbb{R}^n)$, n odd, whose potential V is super-exponentially decreasing. By definition, this means that for every N there is a constant C such that

$$|V(x)| \le Ce^{-N|x|}.$$

Let $R(k) = (H - k^2)^{-1}$ be the resolvent of H, initially defined for k in the upper half plane. Let $R_V(k)$ denote the weighted resolvent

$$R_V(k) = V^{\frac{1}{2}} R(k) |V|^{\frac{1}{2}}.$$

When V is super-exponentially decreasing and n is odd, $R_V(k)$ has a compact operator valued meromorphic continuation to the entire complex plane. Resonances are poles in this meromorphic continuation.

We are interested in upper bounds for the counting function

$$n(r) = \#\{\text{resonances} \quad k : |k| < r\}.$$

Here is what is known about the large r behaviour of $n(r)$.

When $n = 1$ and V has compact support

$$n(r) = \frac{2}{\pi}\mathrm{diam}(\mathrm{supp}(V))r + o(r).$$

This result is due to Zworski [Z1]. For some super-exponentially decaying potentials in one dimension there is a comparable result [F]

$$n(r) = Cr^{\rho} + o(r^{\rho}),$$

where ρ is the order of growth of the Fourier transform of V. For a class of radially symmetric potentials in dimensions greater than one Zworski [Z2] proves

$$n(r) = C_n \mathrm{radius}(\mathrm{supp}(V))r^n + o(r^n).$$

This exhausts the examples of Schrödinger operators for which the first term in an asymptotic expansion is known.

For a general compactly supported potential in dimensions greater than one, only upper bounds (of polynomial type) are known:

$$n(r) \leq Cr^n.$$

The first polynomial bound (r^{n+1}) was obtained by Melrose [Me1]. The sharp bound (r^n) was first obtained by Zworski [Z3]. For potentials decreasing like $\exp(-|x|^{1+\epsilon})$, Sá Barreto and Zworski [SZ1] prove

$$n(r) \leq Cr^{(1+1/\epsilon)n}.$$

In we generalize this to super-exponentially decaying potentials. The Fourier transform of a potential in this class is an entire function in $\mathbb{C}^n$. Our bound is given in terms of the growth of the Fourier transform. We will show that if $\widehat{V}(k)$ grows like $\exp(\Phi(|k|))$, then

$$n(r) \leq C\Phi^n(cr).$$

Although polynomial lower bounds have not been established in general, it is known that infinitely many resonances exist. This was shown in three dimensions by Melrose [Me2], and in any odd dimension by Sá Barreto and Zworski [SZ2]. The existence of infinitely many resonances was previously only known under positivity conditions on the potential (see the references in [SZ2]).

There is a substantial literature on closely related problems, involving, for example, metric perturbations of the Laplacian or even dimensions. We mention here only the paper of Vodev [V] which was influential in our thinking. Further references can be found in the review article [Z4] and the book [Me2].

It is worth pointing out that there are many other definitions of resonances in the literature. The theory of dilation and translation analyticity give rise to definitions that do not require the potential to decrease so rapidly.

Meromorphic continuation and the scattering operator

We begin by using standard Birman-Schwinger identities to show that $R_V(k)$ has a meromorphic continuation. Let $R_0(k) = (-\Delta - k^2)^{-1}$ be the free resolvent and define $R_{0V}(k) = V^{\frac{1}{2}} R_0(k) |V|^{\frac{1}{2}}$. For $\operatorname{Im} k > 0$, the resolvent equation

$$R_0(k) - R(k) - R(k) V R_0(k) = 0$$

implies

$$(1 - R_V(k))(1 + R_{0V}(k)) = 1.$$

For $\operatorname{Im} k$ large, the norm of $R_{0V}(k)$ is small, so $1 + R_{0V}(k)$ is invertible and

$$1 - R_V(k) = (1 + R_{0V}(k))^{-1} \qquad ()$$

The operator $R_{0V}(k)$ has an explicit integral kernel given by $V^{\frac{1}{2}}(x) G_0(x, y, k) |V|^{\frac{1}{2}}(y)$, where G_0 is the free Greens function. Using this representation, it is easy to see that in odd dimensions $R_{0V}(k)$ has a compact operator valued analytic continuation to the entire complex plane (except for a pole at zero in dimension 1). Thus, by the meromorphic Fredholm theorem [S], the left side of () defines a meromorphic continuation for $R_V(k)$.

From this formula we can see that the resonances are precisely those values of k for which $R_{0V}(k)$ has an eigenvalue -1. Equivalently, resonances are precisely the zeros of the analytic function $\det(1 + R_{0V}(k))$—provided $R_{0V}(k)$ is trace class. Unfortunately, this only happens when $n = 1$. In higher dimensions, it turns out that $R_{0V}^p(k)$ for $p > n/2$ is trace class. If -1 is an eigenvalue for $R_{0V}(k)$, then ± 1 is an eigenvalue for $R_{0V}^p(k)$. Thus the set of resonances is contained in the set of zeros of the function $\det(1 \pm R_{0V}^p(k))$ for $p > n/2$ and $\pm(-1)^p = 1$. This function is entire, except for poles arising from eigenvalues. One can therefore estimate the number of resonances by estimating the growth of this function. This approach is used in previous work.

Our approach is to first multiply $1 + R_{0V}(k)$ by a suitable invertible operator and then take the determinant. For k in the lower half plane, $-k$ is in the upper half plane, and so by () the operator $1 + R_{0V}(-k)$ is invertible (with inverse $1 - R_V(-k)$) except at the finitely places where k^2 is an eigenvalue of H. Define

$$Q(k) = (1 + R_{0V}(-k))^{-1}(1 + R_{0V}(k))$$

We will show that the determinant of $Q(k)$ is well defined. Then resonances are precisely the zeros of $\det Q(k)$ in the lower half plane. The disadvantage of our regularization is that $\det Q(k)$ is not entire, but has poles in the upper half plane. (Clearly $-k$ is pole whenever k is a zero.) This makes it necessary to include an estimate of the scattering phase in our proof.

In fact, we will show presently that $\det Q(k) = \det S(-k)$ where $S(k)$ is the scattering operator, confirming that resonances are precisely equal to scattering poles. This equivalence is well known, but we could not resist giving this simple proof.

Begin with the classical Green's function identity

$$G_0(x,y,k) - G_0(x,y,-k) = c(k) \int_{S^{n-1}} e^{ik\langle \omega, x-y\rangle} d\omega, \qquad ()$$

where

$$c(k) = \frac{i\pi k^{n-2}}{(2\pi)^n}.$$

This can be proven using the representation of G_0 as a Fourier transform [see Me2], or by applying Green's theorem to the functions $f(z) = G_0(x,z,k)$, $g(z) = G_0(z,y,-k)$ in a sphere of radius r and letting r tend to infinity.

The identity () for real k can be written as an operator equation. Let π_k denote the operator that takes a function on $\mathbb{R}^n$ to its Fourier transform restricted to a sphere of radius k. Then () can be written

$$R_0(k) = R_0(-k) + c(k)\pi_k^*\pi_k.$$

This has to be interpreted as an equation involving operators between Besov spaces, since none of the operators are bounded on $L^2(\mathbb{R}^n)$ for real k. However, if we multiply on the left by $V^{\frac{1}{2}}$ and on the right by $|V|^{\frac{1}{2}}$ then we do obtain an equation involving operators on $L^2(\mathbb{R}^n)$, namely

$$R_{0V}(k) = R_{0V}(-k) + c(k)F_V^T(k)F_{|V|}(k) \qquad ()$$

where the operator $F_{|V|}(k) : L^2(\mathbb{R}^n) \to L^2(S^{n-1})$ is the formal product $\pi_k|V|^{\frac{1}{2}}$ is given by

$$\left(F_{|V|}(k)\psi\right)(\omega) = \int_{\mathbb{R}^n} e^{ik\langle x,\omega\rangle}|V|^{\frac{1}{2}}(x)\psi(x)d^nx,$$

and the operator $F_V^T(k) : L^2(S^{n-1}) \to L^2(\mathbb{R}^n)$ is the formal product $V^{\frac{1}{2}}\pi_k^*$ given by

$$\left(F_V^T(k)\phi\right)(x) = V^{\frac{1}{2}}(x)\int_{S^{n-1}} e^{ik\langle x,\omega\rangle}\phi(\omega)d\omega$$

Since all the operators in the equation () have analytic continuations to complex k, the equation remains valid for all $k \in \mathbb{C}$.

We now return to $Q(k)$. Using () this operator can be rewritten

$$\begin{aligned} Q(k) &= (1 + R_{0V}(-k))^{-1}(1 + R_{0V}(k)) \\ &= (1 + R_{0V}(-k))^{-1}(1 + R_{0V}(-k) + c(k)F_V^T(k)F_{|V|}(k)) \\ &= 1 + c(k)(1 + R_{0V}(-k))^{-1}F_V^T(k)F_{|V|}(k)) \\ &= 1 + c(k)(1 - R_V(-k))F_V^T(k)F_{|V|}(k) \end{aligned}$$

It follows from the estimates on singular values below that the second term on the left side is trace class, so we may take the determinant. Using the identity $\det(1+AB) = \det(1+BA)$ gives

$$\det Q(k) = \det(1 + c(k)F_{|V|}(k)(1 - R_V(-k))F_V^T(k)).$$

The operators on the left are now operators on $L^2(S^{n-1})$. Now

$$c(k)F_{|V|}(k)(1 - R_V(-k))F_V^T(k) = c(k)\pi_k(V - VR(-k)V)\pi_k^* = T(-k), \qquad ()$$

is exactly the expression from stationary scattering theory for the T matrix. Thus

$$\det Q(k) = \det(1 + T(-k)) = \det S(-k),$$

as claimed.

Upper bounds

We now state our main theorem.

Theorem *Suppose that V is a super-exponentially decaying potential with*

$$\widehat{V}(z) \leq Ce^{\Phi(|z|)}$$

Then

$$n(r) \leq C\Phi^n(cr) + O(\Phi^{n-1}(cr))$$

for some constants c and C.

The proof will be broken up into a series of lemmas. We begin with a simple estimate.

Lemma *Let $\phi(k) = \det(1 + T(k))$ where $T(k)$ is a trace-class operator valued analytic function in the closed upper half plane, where $1 + T(s)$ is unitary for $s \in \mathbb{R}$, and where $T(0) = 0$. Let $n(t)$ denote the number of zeros of $\phi(k)$ in a half disk in the upper half plane of radius t. Define*

$$N(r) = \int_0^r \frac{n(t)}{t}dt.$$

Then

$$N(r) \leq \frac{1}{2\pi}\int_0^r t^{-1}\int_{-t}^t \|T'(s)\|_1 ds dt + \frac{1}{2\pi}\int_0^\pi \ln|\phi(re^{i\theta})|d\theta$$

Here $\|\cdot\|_1$ denotes the trace norm.

Proof: Integrating along a contour enclosing the half disk, we have

$$\begin{aligned}
n(t) &= \frac{1}{2\pi i}\oint \frac{\phi'(k)}{\phi(k)}dk \\
&= \frac{1}{2\pi}\,\mathrm{Im}\int_{-t}^{t}\frac{\phi'(s)}{\phi(s)}ds + \frac{1}{2\pi}\int_0^{\pi} t\frac{d}{dt}\ln|\phi(te^{i\theta})|d\theta \\
&\le \frac{1}{2\pi}\int_{-t}^{t}|\phi'(s)|ds + \frac{1}{2\pi}\int_0^{\pi} t\frac{d}{dt}\ln|\phi(te^{i\theta})|d\theta
\end{aligned}$$

We used the fact that $|\phi(s)| = 1$ for real s. Dividing by t and integrating, we find

$$N(r) \le \frac{1}{2\pi}\int_0^r t^{-1}\int_{-t}^{t}|\phi'(s)|dsdt + \frac{1}{2\pi}\int_0^{\pi}\ln|\phi(re^{i\theta})|d\theta$$

We used $\phi(0) = 1$ to evaluate the second term. Since $\phi'(s) = \phi(s)\mathrm{tr}((1+T(s))^{-1}T'(s))$, and $|\phi(s)| = \|(1+T(s))^{-1}\| = 1$ for real s, it follows that for real s

$$\begin{aligned}
|\phi'(s)| &\le \|((1+T(s))^{-1}T'(s)\|_1 \\
&\le \|(1+T(s))^{-1}\|\|T'(s)\|_1 \\
&= \|T'(s)\|_1.
\end{aligned}$$

This completes the proof. □

To apply this lemma, we must estimate the trace norm of T' along the real axis, and the growth of $\phi = \det(1+T)$ in the upper half plane when T is given by ().

Lemma *Let V be a super-exponentially decaying potential and let $T(k)$ be given by (). Then for $s \in \mathbb{R}$*

$$\|T'(s)\|_1 \le C|s|^{n-2}$$

Proof: The operator $T(s)$ is a product $c(s)F_{|V|}(s)(1 - R_V(-s))F_V^T(s)$. We estimate each term and its derivative. To begin, we have

$$\begin{aligned}
|c(s)| &\le C|s|^{n-2} \\
|c'(s)| &\le C|s|^{n-3}
\end{aligned}$$

It follows from the representation () that

$$\begin{aligned}
\|1 - R_V(-s)\| &\le C \\
\|R_V'(-s)\| &\le C
\end{aligned}$$

Using the explicit integral kernels for $F_V^T(s)$ and $F_{|V|}(s)$ it easy to estimate the Hilbert-Schmidt norms

$$\begin{aligned}
\|F_V^T(s)\|_2^2 &= \int_{S^{n-1}}\int_{\mathbb{R}^n}|e^{is\langle\omega,x\rangle}V^{\frac{1}{2}}(x)|^2dxd\omega \le C \\
\|F_V^{T\prime}(s)\|_2^2 &= \int_{S^{n-1}}\int_{\mathbb{R}^n}|i\langle\omega,x\rangle e^{is\langle\omega,x\rangle}V^{\frac{1}{2}}(x)|^2dxd\omega \le C.
\end{aligned}$$

The same estimates hold for $\|F_{|V|}(s)\|_2$ and $\|F'_{|V|}(s)\|_2$. The proof is completed by using the Leibnitz rule to write $T'(s)$, and the estimate

$$\|AB\|_1 \le \|A\|_2^{\frac{1}{2}}\|B\|_2^{\frac{1}{2}}$$

□

It remains to estimate the growth of $\phi(k)$ for complex k.

Lemma *Suppose V is a super-exponentially decreasing potential, and let $T(k)$ be given by (). Then for k in the upper half plane, $T(k)$ is trace class. Let $\phi(k) = \det(1 + T(k))$. Suppose that the Fourier transform of V satisfies the growth estimate*

$$\widehat{V}(z) \leq Ce^{\Phi(|z|)}$$

for some positive, increasing function $\Phi(x)$. Then

$$|\phi(k)| \leq \exp(\delta^{-(n-1)}\Phi^n((2+\epsilon)|k|) + O(\Phi^{n-1}((2+\epsilon)|k|)))$$

for any $\epsilon > 0$, and some constant C.

Proof: We will use Weyl's estimate

$$|\phi(k)| = |\det(1 + T(k))| \leq \prod_j \left(1 + \mu_j(T(k))\right)$$

and therefore must estimate the singular values of $T(k)$. Using the estimate

$$\mu_j(AB) \leq \|A\|\mu_j(B)$$

and

$$\mu_j(AB) = \mu_j(BA)$$

(twice) we find

$$\begin{aligned}\mu_j(T(k)) &\leq \|c(-k)(1 - R_V(k)\|\mu_j(F_V^T(-k)F_{|V|}(-k)) \\ &\leq C|k|^{n-2}\mu_j(\mathbf{V}_k).\end{aligned} \qquad ()$$

Where the operator $\mathbf{V}_k = F_{|V|}(-k)F_V^T(-k)$ is an integral operator on $L^2(S^{n-1})$ with integral kernel $\widehat{V}(-k(\omega - \omega'))$. To estimate the singular values we will use the following bound without proof. (It follows from the analyticity of $\widehat{V}$.) Let L_ω denote the positive Laplacian on S^{n-1} in the variables ω. Then for any $\epsilon > 0$ there is a constant C such that

$$|L_\omega^{p/2}\widehat{V}(k(\omega - \omega'))| \leq C^p p! e^{\Phi((2+\epsilon)|k|)}$$

Summing the Taylor expansion for the exponential, this gives, for $\delta < C^{-1}$,

$$|e^{\delta L_\omega^{\frac{1}{2}}}\widehat{V}(k(\omega - \omega'))| \leq (1 - \delta C)^{-1}e^{\Phi((2+\epsilon)|k|)}$$

Since the left side is the integral kernel for the operator $e^{\delta L_\omega^{\frac{1}{2}}}\mathbf{V}_k$, this implies that

$$\|e^{\delta L_\omega^{\frac{1}{2}}}\mathbf{V}_k\| \leq Ce^{\Phi((2+\epsilon)|k|)}$$

(The constants C may change from line to line.) Thus

$$\begin{aligned}\mu_j(\mathbf{V}_k) &= \mu_j(e^{-\delta L_\omega^{\frac{1}{2}}} e^{\delta L_\omega^{\frac{1}{2}}} \mathbf{V}_k)\\ &\le \|e^{\delta L_\omega^{\frac{1}{2}}} \mathbf{V}_k\| \mu_j(e^{-\delta L_\omega^{\frac{1}{2}}})\\ &\le C e^{\Phi - \delta j^{1/(n-1)}}\end{aligned}$$

Here $\Phi = \Phi((2+\epsilon)|k|)$. Using (), we get the same bound for $\mu_j(T(k))$, if we increase ϵ slightly. Thus $T(k)$ is trace class, and

$$|\det(1+T(k))| \le \prod (1 + C e^{\Phi - \delta j^{1/(n-1)}}).$$

This product is easily estimated by breaking it into two pieces. For $j \le (\Phi/\delta)^{n-1}$ we obtain

$$\begin{aligned}\prod_{j \le (\Phi/\delta)^{n-1}} (1 + C e^{\Phi - \delta j^{1/(n-1)}}) &\le \prod_{j \le (\Phi/\delta)^{n-1}} (C+1) e^{\Phi - \delta j^{1/(n-1)}}\\ &= (C+1)^{(\Phi/\delta)^{n-1}} e^{\Phi^n/\delta^{n-1}} \exp\left(\sum_{j=1}^{(\Phi/\delta)^{n-1}} -\delta j^{1/(n-1)}\right)\\ &= e^{\Phi^n/\delta^{n-1} + O(\Phi^{n-1})}\end{aligned}$$

For $j > (\Phi/\delta)^{n-1}$ we have

$$\begin{aligned}\prod_{j > (\Phi/\delta)^{n-1}} (1 + C e^{\Phi - \delta j^{1/(n-1)}}) &\le \exp\left(\sum_{j > (\Phi/\delta)^{n-1}} C e^{\Phi - \delta j^{1/(n-1)}}\right)\\ &\le \exp\left(C e^{\Phi} \sum_{j > (\Phi/\delta)^{n-1}} e^{-\delta j^{1/(n-1)}}\right)\end{aligned}$$

The sum appearing in this formula can be estimated by an integral.

$$\begin{aligned}\sum_{j > (\Phi/\delta)^{n-1}} e^{-\delta j^{1/(n-1)}} &\le e^{-\delta(\Phi/\delta)} + \int_{(\Phi/\delta)^{n-1}}^{\infty} e^{-\delta x^{1/(n-1)}} dx\\ &\le e^{-\Phi} + C e^{-\Phi} \Phi^{n-2}\end{aligned}$$

Thus the product for large j satisfies the bound

$$\prod_{j > (\Phi/\delta)^{n-1}} (1 + C e^{\Phi - \delta j^{1/(n-1)}}) \le e^{C \Phi^{n-2}}.$$

Combining the estimates for small and large j completes the proof.

□

We can now prove the main theorem.

Proof of : Let $T(k)$ be given by (). We must count the zeros of $\phi(k) = \det(1 + T(k))$ in a half disk of radius r in the upper half plane. The operator $T(k)$ is analytic in the upper half plane, except for possibly finitely many poles, which won't affect the counting function. Since $1 + T(k) = S(k)$ is the scattering operator, we know that $1+T(k)$ is unitary for k real, and that $T(0)=0$. Therefore applies. Using to estimate $\|T'(k)\|_1$ and to estimate $\ln|\phi(k)|$ yields

$$
\begin{aligned}
&N(r) \\
&\quad \le C\int_0^r t^{-1}\int_{-t}^{t}|s|^{n-2}ds dt + (2\pi)^{-1}\int_0^{\pi}\left(\delta^{-(n-1)}\Phi^n((2+\epsilon)r) + O(\Phi^{n-1}((2+\epsilon)r))\right) d\theta \\
&\quad \le Cr^{n-1} + 2^{-1}\delta^{-(n-1)}\Phi^n((2+\epsilon)r) + O(\Phi^{n-1}((2+\epsilon)r))
\end{aligned}
$$

By looking at $\widehat{V}$ along the imaginary axis, we see that Φ must grow at least as fast as r. (For compactly supported V's we have $\Phi(r)=Cr$.) Thus we can ignore the first term. This gives

$$
N(r) \le 2^{-1}\delta^{-(n-1)}\Phi^n((2+\epsilon)r) + O(\Phi^{n-1}((2+\epsilon)r))
$$

To get an estimate on $n(r)$ from this, note that for any $s>1$, since $n(r)$ is monotone,

$$
\begin{aligned}
n(r) &= n(r)(\ln s)^{-1}\int_r^{sr}\frac{1}{t}dt \\
&\le (\ln s)^{-1}\int_r^{sr}\frac{n(t)}{t}dt \\
&= (\ln s)^{-1}(N(sr)-N(r)) \\
&\le (\ln s)^{-1}N(sr).
\end{aligned}
$$

This completes the proof. $\square$

References

[F] Richard Froese, *Asymptotic distribution of resonances in one dimension,* J. Diff. Eqns. **137** no. 2 (1997) 251–272.

[Me1] Richard Melrose, *Polynomial bound on the number of scattering poles,* J. Functional Analysis, **53**, 1983, 287–303

[Me2] Richard Melrose, *Geometric scattering theory,* Cambridge University Press, 1995

[S] Barry Simon, *Trace Ideals and their Applications,* London Mathematical Society Lecture Note Series 35, Cambridge Univ. Press, UK, 1979

[SZ1] Antônio Sá Barreto and Maciej Zworski, *Existence of resonances in three dimensions,* Commun. Math. Phys. **173** (1995) no. 2 401–415

[SZ2] Antônio Sá Barreto and Maciej Zworski, *Existence of resonances in potential scattering,* Commun. Pure Appl. Math. **49** (1996) no. 12 1271–1280

[Z1] Maciej Zworski, *Distribution of poles for scattering on the real line,* J. of Funct. Anal., **73** (2) , (1987), 277–296

[Z2] Maciej Zworski, *Sharp polynomial bounds on the number of scattering poles of radial potentials,* J. of Funct. Anal., **82** (2) , (1989), 370–403

[Z3] Maciej Zworski, *Sharp polynomial bounds on the number of scattering poles,* Duke Math. Jour, **59** (2) , (1989), 311–323

[Z4] Maciej Zworski, *Counting scattering poles,* Spectral and scattering theory (M. Ikawa, ed.), Marcel Dekker.

[V] Georgi Vodev, *Sharp polynomial bounds on the number of scattering poles of perturbations of the Laplacian,* Comm. Math. Phys. **146**, (1992), 205–216.