Braid subgroup normalisers, commensurators and induced representations

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1. Introduction.

The classical braid groups of E. Artin form an ascending sequence $B_1 \subset B_2 \subset \cdots \subset B_\infty$. These groups, and their representations, play an important rôle in several areas of mathematics and physics. The object of this note is to establish new algebraic information about this sequence of groups. In particular, we determine the normaliser and commensurator of the braid group B_n in B_m , $n \leq m \leq \infty$. They are described in terms of the centraliser, which was recently characterised in [FRZ], and this leads to explicit generators and relations presenting these subgroups. In Section 5 we characterise the commensurator of B_n as a certain stabilizer, under the identification of B_m as the mapping class group of the m-punctured 2-dimensional disk. This identification gives rise to infinitely many natural "geometric" inclusions of B_n in B_m, which we demonstrate to be mutually incommensurable, although they are all conjugate. Indeed, this point of view affords a simple description of the centralisers, normalisers and commensurators of all the geometric braid subgroups (Theorem 5.3). In Section 6 we show that the action of B_m, as the mapping class group, upon appropriate sets of curves, is a transitive and *large* action in the sense of Burger and de la Harpe [BH]. The paper concludes with some applications regarding unitary representations of the braid groups, and those induced by braid subgroups.

Here is a summary of our principal results regarding the B_n sequence; the relevant definitions will follow.

Theorem 1.1. *The normaliser of B_n in B_m, for $1 \leq n \leq m \leq \infty$, is the subgroup of B_m generated by B_n and the centraliser of B_n in B_m.*

Theorem 1.2. *The commensurator of B_n in B_m equals the normaliser, $1 \leq n \leq m \leq \infty$.*

In summary: $Com_{B_m}(B_n) = N_{B_m}(B_n) = \langle B_n, Z_{B_m}(B_n)\rangle$. It is shown, moreover, that B_n is a direct summand of its commensurator.

Corollary 5.8. *$Com_{B_m}(B_n)$ is self-commensurating, that is,*

$$Com_{B_m}(Com_{B_m}(B_n)) = Com_{B_m}(B_n).$$

As a typical application, in Section 7 we consider λ_{B_m/C_n}, the left-regular representation of B_m upon $l^2(B_m/C_n)$, where $C_n = Com_{B_m}(B_n)$.

Theorem 7.1. *λ_{B_m/C_n} is irreducible.*

In the first part of this paper, we use largely algebraic arguments to prove Theorems 1.1 and 1.2. Some of these methods extend to algebraic generalisations of the braid groups, such as the Artin groups (see [Par]). To prove Corollary 5.8 and related results, we use geometric arguments inspired by Thurston's treatment of Teichmüller theory (see [FLP]). These techniques also generalise, for example, to mapping class groups of arbitrary surfaces.

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Definitions: Consider a subgroup G of a group H. The *centraliser* $Z_H(G)$ is the set of all $h \in H$ such that $gh = hg$ for all $g \in G$; as a special case, $Z(H) := Z_H(H)$ is the *centre* of H. The *normaliser* $N_H(G)$ is the subgroup of all $h \in H$ such that $h^{-1}Gh = G$. Two subgroups G_1 and G_2 of H are said to be *commensurable* if $G_1 \cap G_2$ has finite index in both G_1 and G_2. The *commensurator* $Com_H(G)$ of G in H is the subgroup of all $h \in H$ such that $h^{-1}Gh$ and G are commensurable. Clearly $Z(H) \subset Z_H(G) \subset N_H(G) \subset Com_H(G)$ and also $G \subset N_H(G)$.

The reader may refer to [Art], [Bir] or [BZ] for basic reference on the braid groups, but to fix notation a quick summary is in order. The group B_∞ has an abstract presentation with the countable set of generators $\sigma_1, \sigma_2, \dots$ and relations:

$$\sigma_i\sigma_j = \sigma_j\sigma_i, \quad |i-j| > 1, \qquad \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}.$$

For $n < \infty$, we may consider B_n to be the subgroup of B_∞ generated by $\sigma_1, \dots, \sigma_{n-1}$; the above relations, restricted to $i, j < n$, give a presentation of B_n.

Elements of B_∞ have a well-known interpretation as strings in 3-space, with concatenation being the group operation. Specifically, model a slab of 3-space by the product $\mathbf{C} \times \mathbf{I}$ of the complex plane and the interval $\mathbf{I} = [0, 1]$. A braid is then taken to be a countable set of strings in $\mathbf{C} \times \mathbf{I}$, monotone in the $\mathbf{I}$ direction, connecting the set of points $\{(1, 0), (2, 0), (3, 0), \dots\}$ to $\{(1, 1), (2, 1), (3, 1) \dots\}$, and with the property that all but a finite number of strings are straight, i. e. of the form $\{n\} \times \mathbf{I}$. Two such geometric braids are considered equivalent if one can be deformed into the other by an isotopy of $\mathbf{C} \times \mathbf{I}$ which has compact support

The initial point $(n, 0)$ defines the *index* $= n$ of a string. The generator σ_i is represented by the braid in which the string of index i crosses under the string of index $i + 1$ in the customary planar picture (perpendicular to the $\mathbf{I}$ direction) and the other strings are straight. B_n can be regarded as the set of braids which have a planar picture in which no string of index $> n$ has crossings with any other string.

A braid β determines a permutation p_β, so that the string of index $p_\beta(n)$ ends at the point $(n, 1) \in \mathbf{C} \times \mathbf{I}$. This defines a homomorphism $B_m \to S_m$, the symmetric group on m letters. Algebraically, this can be considered as introducing the further relations $\sigma_i^2 = 1$. The kernel of this homomorphism is the *pure* or *coloured* braid group P_m.

The centre $Z(B_m)$, and also $Z(P_m)$, are well-known [Chow] to be infinite cyclic for $1 < m < \infty$, generated by the braid which represents a full twist of the strings (or half-twist, in the case of B_2). On the other hand, $Z(B_1)$ is trivial and, since central elements of B_m are no longer central in B_{m+1}, we conclude that $Z(B_\infty)$ is also trivial.

2. Proof of Theorems 1.1 and 1.2.

Some lemmas and definitions will be useful. The first lemma was proven (independently) in [FRZ] and [Gur].

Lemma 2.1. *Suppose β is any braid and σ_i is one of the standard braid generators. If β commutes with some nonzero power σ_i^r, then β commutes with σ_i.* □

Lemma 2.2. *Suppose one has subgroups $F < G < H$ and $h \in Com_H(G)$. Then $G \cap h^{-1}Fh$ has finite index in $h^{-1}Fh$. In particular, $G \cap h^{-1}Fh \neq 1$ if F is infinite.*

proof: It is easy to verify that for any subgroups A, B, C of a group, with $B < C$ the index of subgroups satisfies the inequality $[C : B] \geq [C \cap A : B \cap A]$. In particular, with $C = h^{-1}Gh, B = G \cap h^{-1}Gh, A = h^{-1}Fh$ we have

$$\infty > [h^{-1}Gh : G \cap h^{-1}Gh] \geq [h^{-1}Gh \cap h^{-1}Fh : G \cap h^{-1}Gh \cap h^{-1}Fh] = [h^{-1}Fh : G \cap h^{-1}Fh].$$

□

In the rest of this section, we fix an integer m and define the following subgroups of B_m, for $1 \leq n \leq m$.

B_m^n = the subgroup of braids whose permutation takes $\{1, \dots, n\}$ to itself (setwise).

Note that B_n is a subgroup of B_m^n, not normal, in general.

E_m^n = the subgroup of B_m^n whose first n strings form the identity of B_n.

Lemma 2.3. *$B_n \cap E_m^n = \{1\}$. Also B_n normalises E_m^n, that is, for any $\beta \in B_n, \beta^{-1}E_m^n\beta = E_m^n$.*

The proof is left to the reader.

Lemma 2.4. *If $\gamma \in B_m^n$, then one may write $\gamma = \beta\epsilon$ where $\beta \in B_n$ and $\epsilon \in E_m^n$.*

Proof: this is a variation of the well-known "combing" argument of Artin [Art]. Consider only the first n strings of γ and let β be the element of B_n represented by these strings. Of course, by the inclusion, we can also consider $\beta \in B_m$. Then $\gamma = \beta\beta^{-1}\gamma$ and $\epsilon = \beta^{-1}\gamma$ is the required element of E_m^n. The expression is actually unique, although this fact will not be needed. □

We now turn to the proof of Theorems 1.1 and 1.2. The theorems are obvious for $n = 1$ and $n = m$, and the case $m = \infty$ follows directly (by taking unions) from the finite cases, so we may assume $1 < n < m < \infty$. As already observed, $Com_{B_m}(B_n) \supset N_{B_m}(B_n) \supset$

$\langle B_n, Z_{B_m}(B_n)\rangle$, so we need only show that $Com_{B_m}(B_n) \subset \langle B_n, Z_{B_m}(B_n)\rangle$ to prove both theorems simultaneously. Let γ denote an arbitrary element of $Com_{B_m}(B_n)$.

(1) $\gamma \in B_m^n$

Otherwise we could find strings of γ in $\mathbf{C} \times \mathbf{I}$ which connect a pair $\{(j,0),(j+1,0)\}$, with $j+1 \leq n$, to (say) $\{(k,1),(l,1)\}$ for some $k > n$. For this value of j, let $F = \{\sigma_j^r\}$ be the infinite cyclic subgroup of B_n generated by the braid generator σ_j. Let f be the integer-valued function on B_m which assigns to a braid β the number $f(\beta) =$ the algebraic number of crossings of the string of index k with the string of index l. By inspection, $f(\gamma^{-1}\sigma_j^r\gamma) = r$. On the other hand f is identically zero on elements of B_n since $k > n$. It follows that B_n intersects $\gamma^{-1}F\gamma$ in just the identity. This clearly contradicts Lemma 2.2 (with $G = B_n, h = \gamma$) and establishes (1).

Now Lemma 2.4 provides an expression $\gamma = \beta\epsilon$ with $\beta \in B_m$ and $\epsilon \in E_m^n$. Note that ϵ also belongs to $Com_{B_m}(B_n)$, since β and γ do.

(2) ϵ commutes with all elements of B_n.

To see this, fix $i < n$ and consider, for all integers r, commutators of the form $\epsilon^{-1}\sigma_i^{-r}\epsilon\sigma_i^r$. These commutators lie in E_m^n, since B_n normalises E_m^n. On the other hand, again with the help of Lemma 2.2, since $\epsilon \in Com_{B_m}(B_n)$, we must have $\epsilon^{-1}\sigma_i^{-r}\epsilon \in B_n$ for some nonzero r, and for this r, the commutator also lies in B_n. Since $B_n \cap E_m^n = \{1\}$ we conclude that ϵ commutes with σ_i^r. By Lemma 2.1, ϵ commutes with σ_i. Since this holds for all $i < n$ we have established (2) and the proof of Theorems 1.1 and 1.2 is now complete. □

3. Structure of the normaliser and commensurator.

The structure of the centraliser of B_n in B_m, determined recently in [FRZ], has the following topological description. Let $k = m - n + 1$ and let B_k^1 denote the subgroup of k-braids whose permutation preserves 1. (Alternatively, B_k^1 can be regarded as the $(k-1)$- string braid group of an annulus.) Replace the first string of such a braid by n parallel strings lying on a ribbon, with possibly some number of twists (central elements of B_n). The braids resulting from this operation are exactly the centraliser of B_n. Thus $Z_{B_m}(B_n)$ is isomorphic to a direct product $B_k^1 \times \mathbf{Z}$, the infinite cyclic factor being the centre of B_n. If, instead of just a twist, we put on the ribbon an arbitrary n-braid, we get a geometric picture of a typical element of the normaliser (and commensurator) of B_n in B_m, according to Theorems 1 and 2. It follows that the normaliser actually splits as a direct product in a natural way:

Theorem 3.1. *The normaliser (and commensurator) of B_n in B_m is isomorphic with the direct product $B_n \times B_k^1$, where $k = m - n + 1$.* □

To describe $N_{B_m}(B_n)$ by generators and relations, we first recall the structure of B_k^1 as determined by Chow [Chow]. Let $\sigma_1, \ldots, \sigma_{k-1}$ denote the standard generators of B_k. The subgroup B_k^1 of B_k is generated by $\sigma_2, \ldots, \sigma_{k-1}$, together with elements $a_2, \ldots, a_k$ defined by

$$a_i := \sigma_1^{-1}\sigma_2^{-1}\cdots\sigma_{i-2}^{-1}\sigma_{i-1}^2\sigma_{i-2}\cdots\sigma_2\sigma_1.$$

These generators satisfy the usual braid relations:

$$\begin{aligned}\sigma_i\sigma_j &= \sigma_j\sigma_i, \quad |i-j|>1\\ \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1}\end{aligned}$$

as well as the following, for $i = 2, \ldots, k$:

$$\begin{aligned}\sigma_i a_j \sigma_i^{-1} &= a_j, \quad j \neq i, i+1\\ \sigma_i a_i \sigma_i^{-1} &= a_{i+1}\\ \sigma_i a_{i+1}\sigma_i^{-1} &= a_{i+1}^{-1}a_i a_{i+1}.\end{aligned}$$

In fact these are *defining* relations for B_k^1. Applying this to our situation, for each $i = 1, \ldots, m-n$, let α_{n+i} be the m-braid resulting from replacing the first string of the k-braid a_i, defined above, by n parallel strings, as described above. Specifically,

$$\alpha_{n+i} = (\sigma_n^{-1}\sigma_{n+1}^{-1}\cdots\sigma_{n+i-2}^{-1}\sigma_{n+i-1})(\sigma_{n-1}^{-1}\sigma_n^{-1}\cdots\sigma_{n+i-3}^{-1}\sigma_{n+i-2})\cdots(\sigma_1^{-1}\sigma_2^{-1}\cdots\sigma_{i-1}^{-1}\sigma_i)\times$$
$$\times(\sigma_i\sigma_{i-1}\cdots\sigma_1)(\sigma_{i+1}\sigma_i\cdots\sigma_2)\cdots(\sigma_{n+i-1}\sigma_{n+i-2}\cdots\sigma_n).$$

Note that in the copy of B_k^1 in Theorem 3.1, the subscripts of the generators must be shifted by adding $n-1$.

Theorem 3.2. *The normaliser $N_{B_m}(B_n), n < m < \infty$, has generators:*

$$\sigma_1, \ldots, \sigma_{n-1}, \sigma_{n+1}, \ldots, \sigma_m, \alpha_{n+1}, \ldots, \alpha_m$$

and defining relations:

$$\begin{aligned}\sigma_i\sigma_j &= \sigma_j\sigma_i, && |i-j|>1\\ \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1}\\ \sigma_i\alpha_j\sigma_i^{-1} &= \alpha_j, && j\neq i, i+1\\ \sigma_i\alpha_i\sigma_i^{-1} &= \alpha_{i+1}\\ \sigma_i\alpha_{i+1}\sigma_i^{-1} &= \alpha_{i+1}^{-1}\alpha_i\alpha_{i+1}.\end{aligned}$$

(Subscripts ranging over all values for which the symbols are in the list of generators.)

Proof: By Theorem 3.1, one gets a presentation by including all generators of B_n and B_k^1, their defining relations, and relations which say the generators of B_n commute with those of B_k^1. These latter are given by the first and third set of relations, with $i < n < j$. □

Finally, we turn to an alternate geometric description of the centraliser and normaliser of B_n in B_m. In the complex plane $\mathbf{C}$ consider the circle $\Gamma(n) := \{z \in \mathbf{C}; |z| = n + 1/2\}$.

Theorem 3.3. *The centraliser $Z_{B_m}(B_n)$ consists of braids in $\mathbf{C} \times I$ which (possibly after a braid isotopy) are disjoint from the annulus $\Gamma(n) \times \mathbf{I}$, and such that the part of the braid enclosed by $\Gamma(n) \times \mathbf{I}$ is in the centre of B_n, i. e. some power of a twist. The normaliser (= commensurator) consists of arbitrary braids which are, up to isotopy, disjoint from $\Gamma(n) \times \mathbf{I}$.*

Proof: For the normaliser, it is clear that braids disjoint from $\Gamma(n) \times \mathbf{I}$ are contained in $N_{B_m}(B_n)$. On the other hand, each generator of $N_{B_m}(B_n)$, as listed in Theorem 3.2, is clearly representable as a braid disjoint from $\Gamma(n) \times \mathbf{I}$, so the opposite inclusion also holds. A similar argument applies to the centraliser. □

4. Pure braids.

Since the pure braid group P_n is finite index in B_n, their commensurators coincide and we have the following.

Theorem 4.1. *The commensurator and normaliser of P_n in B_m is generated by B_n and the centraliser of B_n in B_m. The commensurator and normaliser of P_n in P_m is generated by P_n and the centraliser of P_n in P_m. This centraliser is just the intersection of P_m with the centraliser of B_n in B_m.* □

5. Geometric subgroups, stabilisers, and commensurability.

We now turn to the interpretation of braids as mapping class groups. Further details can be found in [Bir], Chapter 4 and [FLP], Exposé 2.

Consider a disk D^2 with a specified finite subset $M \subset int(D^2)$, called "punctures", $|M| = m < \infty$. Define $Diff(D^2, M) =$ the group of diffeomorphisms $f : D^2 \to D^2$ such that $f(M) = M$ and f is the identity on some neighbourhood of ∂D^2. This group can be topologised in a natural way, and the set of isotopy classes $\pi_0(Diff(D^2, M))$ forms a (discrete) group. For purposes of studying B_m we can take as our standard the set $M = \{1, \dots, m\}$ within the disk $D^2 = \{z \in \mathbf{C}; |z| \leq m + 1/2\}$. Then we have a canonical isomorphism with the (algebraically defined) braid group:

$$B_m \cong \pi_0(Diff(D^2, M)).$$

Under this identification, the generator $\sigma_j \in B_m$ corresponds to a diffeomorphism of D^2 which is the identity outside a small neighbourhood of the straight line interval $[j, j+1]$, and within that neighbourhood it performs a "half-twist" (in say a clockwise direction) interchanging the punctures j and $j+1$. Then the previously-defined braid subgroup $B_n \subset B_m$ corresponds exactly to those diffeomorphisms which (up to isotopy) are the identity outside the circle $\Gamma(n) = \{z \in \mathbf{C}; |z| = n + 1/2\}$.

More generally, we can define "geometric" braid subgroups of B_m in a natural way, corresponding to inclusion of arbitrary subdisks. These will be parametrized by the collection $\mathcal{C}$ of smooth simple closed curves in D^2 which are disjoint from the set M. Curves which are isotopic in $D^2 - M$ are regarded as equal in $\mathcal{C}$, and we will abuse notation by using the same symbol for a curve and its isotopy class. For $\Gamma \in \mathcal{C}$, let $D_\Gamma \subset D^2$ be the disk with $\partial D_\Gamma = \Gamma$ and let $M_\Gamma := M \cup D_\Gamma$. Then define the braid group B_Γ by

$$B_\Gamma := \pi_0(Diff(D_\Gamma, M_\Gamma)).$$

By the obvious natural inclusion (a diffeomorphism of D_Γ extends by the identity to D^2)

$$\pi_0(Diff(D_\Gamma, M_\Gamma)) \subset \pi_0(Diff(D^2, M))$$

we can regard B_Γ as a subgroup of B_m, which we will call a *geometric braid subgroup*.

An equivalent definition is

$$B_\Gamma = Fix(D^2 - D_\Gamma),$$

where we use the notation $Fix(X)$, for $X \subset D^2$, to denote the stabiliser of X. That is, $Fix(X)$ is the subgroup of $\pi_0(Diff(D^2, M))$ of (classes of) diffeomorphisms h such that $h(x) = x$ for all $x \in X$. We note that for reasonable subsets X this is the same as classes of diffeomorphisms fixed on a neighbourhood of X, a convenient assumption to ensure smoothness of functions extended by the identity.

Clearly B_Γ is abstractly isomorphic with the braid group B_n, where $n = |M_\Gamma|$ is the number of punctures enclosed by Γ (we call this the *type* of Γ). But it may be embedded in B_m differently from the standard $B_n = B_{\Gamma(n)}$, even if exactly the same punctures are enclosed by the two curves Γ and $\Gamma(n)$, as we will see below. Define

$$\mathcal{C}_n = \text{isotopy classes of curves of type } n.$$

Lemma 5.1. *Let $\Gamma, \Gamma' \in \mathcal{C}$. Then there is a homeomorphism $h \in Diff(D^2, M)$, with $h(\Gamma) = \Gamma'$ if and only if Γ and Γ' have the same type.*

Proof: Standard plane topology, left to the reader. □

It is also clear that, for $h \in Diff(D^2, M)$, $h(\Gamma) = \Gamma'$ implies $h(D_\Gamma) = D_{\Gamma'}$ and conversely. Moreover, if $g, h \in Diff(D^2, M)$, satisfy $g(\Gamma) = h(\Gamma)$ (setwise) we may assume after an isotopy that g and h agree pointwise on Γ. So for curves, being stabilised setwise and pointwise are equivalent.

Using the observation that $Fix(h(X)) = h(Fix(X))h^{-1}$ we conclude the following

Proposition 5.2. *If Γ and Γ' are curves of the same type, then B_Γ is conjugate to $B_{\Gamma'}$ in B_m, and conversely.* □

In other words, each $\mathcal{C}_n$, $0 \le n \le m$, is an orbit under the action of $\pi_0(Diff(D^2, M))$ on $\mathcal{C}$, and represents exactly a conjugacy class of geometric subgroups. Note that $\mathcal{C}_n$ is an infinite set for each type $1 < n < m$ (we will call curves of such type n *generic*). The three non-generic conjugacy classes are finite: $|\mathcal{C}_0| = 1$; $|\mathcal{C}_1| = m$; $|\mathcal{C}_m| = 1$. We can regard the sequence $B_0 \subset B_1 \subset \ldots \subset B_m$ as a complete list of representatives, one from each conjugacy class of geometric subgroups.

Theorem 5.3. *For each geometric subgroup B_Γ, $\Gamma \in \mathcal{C}$, its centraliser, normaliser and commensurator in $B_m \cong \pi_0(Diff(D^2, M))$ are as follows:*

$$Z_{B_m}(B_\Gamma) = Fix(D_\Gamma),$$

$$N_{B_m}(B_\Gamma) = Com_{B_m}(B_\Gamma) = Fix(\Gamma).$$

Proof. For the special case $\Gamma = \Gamma(n)$ the above are easily seen to be equivalent to Theorem 3.3. The one moot point is that the generator of the centre of B_n can be represented by a Dehn twist (defined below) along a curve parallel to, but outside of, $\Gamma(n)$. So, in fact, this generator may also be considered an element of $Fix(D_{\Gamma(n)})$.

For the general case, if Γ has type n, then $h(\Gamma) = \Gamma(n)$ for some diffeomorphism h of (D^2, M). Then $B_\Gamma = h^{-1}B_n h$ and the centraliser can be computed $Z_{B_m}(B_\Gamma) = h^{-1}Z_{B_m}(B_n)h = h^{-1}Fix(\Gamma(n))h = h^{-1}Fix(h(\Gamma))h = Fix(\Gamma)$. The rest follows similarly, since forming the normaliser and commensurator also commute with conjugation. □

We now turn to the problem of showing that the commensurator of B_n (in fact of any geometric braid subgroup) is self-commensurating. By Theorem 5.3 this amounts to studying $Fix(\Gamma)$ for $\Gamma \in \mathcal{C}$. We will need some lemmas regarding curves and two definitions. A *Dehn twist* T_Γ along a (simple closed) curve $\Gamma \subset D^2 - M$ is a diffeomorphism of (D^2, M) which is the identity outside a small annulus neighbourhood of Γ. If we parametrize this neighbourhood by $(e^{i\theta}, t) \in S^1 \times [0, 1] \cong N(\Gamma)$ the twist may be written

$$T_\Gamma : (e^{i\theta}, t) \longrightarrow (e^{i(\theta+f(t))}, t)$$

where $f : [0, 1] \to [0, 2\pi]$ is a smooth monotonic function with $f(0) = 0$ and $f(1) = 2\pi$. One may also consider "Dehn half-twists" along arcs connecting punctures (σ_j is a prototype). These, of course, generate B_m, whereas Dehn twists along curves always correspond to pure braids.

As in [FLP], we define *intersection number* of two (isotopy classes of) curves in $D^2 - M$. If $\Gamma, \Gamma' \in \mathcal{C}$, define $i(\Gamma, \Gamma')$ to be the minimum number of intersections $|\Gamma \cap \Gamma'|$, over all representatives in their isotopy classes in $D^2 - M$. In particular $i(\Gamma, \Gamma) = 0$ for any (simple) curve $\Gamma \subset D^2 - M$. Note that if two curves have $i(\Gamma, \Gamma') = 0$, then the corresponding Dehn twists commute.

Lemma 5.4. *Let Γ and Γ' be generic curves in $D^2 - M$ with $\Gamma \neq \Gamma'$ in $\mathcal{C}$. Then there is a (generic) curve Γ'' in $D^2 - M$ such that $i(\Gamma', \Gamma'') = 0$ but $i(\Gamma, \Gamma'') \neq 0$.*

Proof. If $i(\Gamma, \Gamma') \neq 0$ we may simply take $\Gamma'' = \Gamma'$. So it suffices to consider Γ and Γ' disjoint – they may be nested or bound disjoint disks. In each case, it is possible to find a puncture $p \in M$ such that Γ separates p from Γ'. Let A denote an arc that connects Γ' to p. Take Γ'' to be a curve that starts on A near Γ', runs around Γ', but parallel to it, until nearly back to A. Then Γ'' runs parallel to A, encircles p, then returns along A to its starting point. By construction, $|\Gamma' \cap \Gamma''| = 0$. On the other hand, in each case, Γ'' encloses punctures which are separated by Γ, so $i(\Gamma, \Gamma'') \neq 0$. □

Lemma 5.5. *Suppose $i(\Gamma, \Gamma') \neq 0$, and $T = T_{\Gamma'}$ is the Dehn twist along Γ'. Then for all powers $k \neq 0$, $T^k(\Gamma) \neq \Gamma$.*

Proof: Actually, we can be more precise and calculate

$$i(T^k(\Gamma), \Gamma) = |k| i(\Gamma, \Gamma')^2.$$

This formula follows by construction and by the criterion that two curves have minimum intersections ($|\Gamma \cup \Gamma'| = i(\Gamma, \Gamma')$) if and only if for each pair of arcs $A \subset \Gamma$ and $A' \subset \Gamma'$ such that A and A' intersect exactly at their endpoints, the curve $A \cup A'$ encloses a puncture. The details are left to the reader. The above formula is actually a special case of a formula of [FLP] (Proposition 1, Appendix to Exposé 4,). Although that formula was proved in a different setting (closed surface rather than punctured disk), similar arguments apply. □

Notice that all curves Γ, Γ' of a given type have conjugate stabilisers: employing Lemma 5.1, $Fix(\Gamma') = Fix(h(\Gamma)) = hFix(\Gamma)h^{-1}$.

Theorem 5.6. *Suppose Γ and $\Gamma' \in \mathcal{C}_n$, $1 < n < m$, then the subgroups $Fix(\Gamma)$ and $Fix(\Gamma')$ of $B_m \cong \pi_0(Diff(D^2, M))$ are commensurable in $\pi_0(Diff(D^2, M))$ if and only $\Gamma = \Gamma'$.*

Proof. Let $G := Fix(\Gamma)$ and $G' := Fix(\Gamma')$ and suppose $\Gamma \neq \Gamma'$. By Lemma 5.4, there is a curve Γ'' satisfying $i(\Gamma', \Gamma'') = 0$ and $i(\Gamma, \Gamma'') \neq 0$. Let $T = T_{\Gamma''}$ be a Dehn twist along Γ''. Consider the infinite set $\{T^k\}$, $k \in \mathbf{Z}$. Because $i(\Gamma', \Gamma'') = 0$, we have $T^k \in G'$ for all k. On the other hand $T^k \in G$ only for $k = 0$ because of Lemma 5.5. Thus we have an infinite subset $\{T^k\}$ of G' which are in different cosets modulo $G \cap G'$. This shows $G \cap G'$ is infinite index in G' and therefore G and G' are noncommensurable. □

Remark: Theorem 5.6 also holds trivially for $n = 0$ or $n = m$, but not for $n = 1$ in B_2. If Γ_1 and Γ_2 are small circles surrounding the points 1 and 2, respectively, we see that $Fix(\Gamma_1)$ and $Fix(\Gamma_2)$ both correspond to the subgroup P_2 of pure braids. Being equal, they are certainly commensurable, although $\Gamma_1 \neq \Gamma_2$.

Theorem 5.7. *For any curve $\Gamma \subset D^2 - M$, $Fix(\Gamma)$ is self-commensurating in $B_m \cong \pi_0(Diff(D^2, M))$.*

Proof. If $G := Fix(\Gamma)$ and $h \in Diff(D^2, M) - G$, we have $h(\Gamma) \neq \Gamma$ and by Theorem 5.6, $hGh^{-1} = Fix(h(\Gamma))$ is not commensurable with G. □

Corollary 5.8. *$Com_{B_m}(Com_{B_m}(B_n)) = Com_{B_m}(B_n)$. More generally, the commensurator of every geometric braid subgroup of B_m is self-commensurating.* □

Theorem 5.9. *Distinct geometric subgroups of B_m are incommensurable. That is, B_Γ and $B_{\Gamma'}$ are commensurable if and only if $B_\Gamma = B_{\Gamma'}$. For generic curves this happens if and only if Γ and Γ' are isotopic in $D^2 - M$.*

Proof: Let us first investigate the case $i(\Gamma, \Gamma') \neq 0$. Consider a Dehn twist $T = T_{\Gamma'}$ and all its powers T^k. These twists T^k all lie in $B_{\Gamma'}$, but none of them, except for $k = 0$, lie in B_Γ. To see this, note that $B_\Gamma \subset Fix(\Gamma)$ and apply Lemma 5.5. Finish the argument as in Theorem 5.6 to conclude B_Γ and $B_{\Gamma'}$ are incommensurable.

It remains to consider the case $i(\Gamma, \Gamma') = 0$, so we may assume the curves Γ and Γ' are disjoint. This breaks into two subcases, according to the situation of the disks D_Γ and $D_{\Gamma'}$ bounded by the curves: they are either disjoint or nested. In the disjoint case, $D_\Gamma \cap D_{\Gamma'} = \emptyset$, we have $B_\Gamma \cap B_{\Gamma'} = Fix(D^2 - D_\Gamma) \cap Fix(D^2 - D_{\Gamma'}) = Fix(D^2) = \{1\}$. Thus B_Γ and $B_{\Gamma'}$ are incommensurable unless both are finite, in which case both are the trivial group, and Γ and Γ' have type 0 or 1.

In the nested case we may assume by symmetry that $D_\Gamma \subset D_{\Gamma'}$. Either $\Gamma = \Gamma'$, and we are done, or there must be a puncture in $D_{\Gamma'} - D_\Gamma$. Since $B_\Gamma \subset B_{\Gamma'}$ we have $B_\Gamma \cup B_{\Gamma'} = B_\Gamma$. As observed earlier, the pair $(B_{\Gamma'}, B_\Gamma)$ is conjugate to $(B_{n'}, B_n)$, with $n' > n$. With a trivial exception, this pair has infinite index, which implies that B_Γ and $B_{\Gamma'}$ are incommensurable. The exception occurs with Γ' of type $n' = 1$ and Γ of type $n = 0$, in which case both B_Γ and $B_{\Gamma'}$ are trivial groups. □

6. Large group actions.

In the previous section we have been discussing the action $(h, \Gamma) \to h(\Gamma)$ of the group $\pi_0(Diff(D^2, M))$ on the set $\mathcal{C}$, and its (compatible) action by conjugation on the set of geometric braid subgroups B_Γ.

For an arbitrary group action $G \times \mathcal{S} \to \mathcal{S}$ of group G on the set $\mathcal{S}$, the following definitions were made in [BH]. The action has *noncommensurable stabilisers* (N.C.S.) if $Fix(s)$ and $Fix(s')$ are incommensurable subgroups of G, for $s \neq s' \in \mathcal{S}$. An action $G \times \mathcal{S} \to \mathcal{S}$ is said to be *large* if for all $s \in \mathcal{S}$, all $Fix(s)$-orbits in $\mathcal{S}$ - $\{s\}$ are infinite. It was pointed out in [BH] that every large action is N.C.S.

Now the action $\pi_0(Diff(D^2, M)) \times \mathcal{C} \to \mathcal{C}$ is neither large nor N.C.S. in general. In an example already cited, the two curves Γ_1, Γ_2 of type 1 in B_2 have equal stabilisers, although $\Gamma_1 \neq \Gamma_2$. However, except for a few such trivial examples, the action does have the above properties. Note that any diffeomorphism of (D^2, M) preserves the *type* of a curve, so $\pi_0(Diff(D^2, M))$ acts on $\mathcal{C}_n$ for each type. According to Lemma 5.1 and Theorem 5.6 we have the following.

Proposition 6.1. *For each $n, 0 \leq n \leq m$, $\pi_0(Diff(D^2, M))$ acts transitively on $\mathcal{C}_n$.* □

Since $\mathcal{C}_n$ is finite for the special cases $n = 0, 1, m$ the action certainly can't be large on those subsets.

Theorem 6.2. *The natural action* $\pi_0(Diff(D^2, M))\times \mathcal{C}_n \to \mathcal{C}_n$ *is a large action if* n *is generic, i. e.* $1 < n < m$.

Proof: We need to show that if Γ and $\Gamma' \in \mathcal{C}_n$ then the set $\{h(\Gamma'); h \in Fix(\Gamma)\}$ is infinite. Use Lemma 5.4 to find a curve $\Gamma'' \subset D^2 - M$ with $i(\Gamma, \Gamma'') = 0$ and $i(\Gamma', \Gamma'') \neq 0$ and let $T = T_{\Gamma''}$ be the corresponding Dehn twist. Then for all k we have $T^k \in Fix(\Gamma)$ but, by use of Lemma 5.5, the curves $T^k(\Gamma')$ are all distinct, so the orbit in question is indeed infinite. □

7. Applications to induced representations.

We show in this section that irreducible unitary representations of the braid groups B_m arise from induction relative to certain subgroups: not the geometric braid subgroups, but rather their commensurators, i. e. stabilisers of a curve in the mapping class group $\pi_0(Diff(D^2, M)) = B_m$. We refer the reader to [Mac] and [BH] for background and details regarding unitary representations and induction. Consider a discrete group G with subgroup G_0. Given a (unitary) representation ρ of G_0, there is a well-defined induced representation $Ind_{G_o}^{G}(\rho)$ of G. In particular, with ρ the trivial representation, we have λ_{G/G_0}, the left regular representation of G in $l^2(G/G_0)$. As formulated in [BH], Theorem 2.1, we have the following results of Mackey:

The representation λ_{G/G_0} is irreducible if and only if $Com_G(G_0) = G_0$, and if this holds, $Ind_{G_o}^{G}(\rho)$ is irreducible for any finite-dimensional irreducible unitary representation ρ of G_0.

If $Com_G(G_i) = G_i$, for subgroups G_0 and G_1, then λ_{G/G_0} and λ_{G/G_1} are unitarily equivalent if and only if G_0 and G_1 are quasicongugate in G, i. e. for some $g \in G$, G_0 and gG_1g^{-1} are commensurable.

This gives an immediate application of Theorem 5.7 and Corollary 5.8:

Theorem 7.1. *Let* B_n *denote the standardly-embedded braid subgroup of* B_m*, with* $1 < n < m \leq \infty$*, and let* $C_n = Com_{B_m}(B_n) = N_{B_m}(B_n) = \langle B_n, Z_{B_m}(B_n)\rangle$*. For each finite-dimensional irreducible unitary representation* ρ *of* C_n*,* $Ind_{C_n}^{B_m}(\rho)$ *is also irreducible. In particular the left regular representation* λ_{B_m/C_n} *of* B_m *is irreducible, whereas* λ_{B_m/B_n} *is reducible.* □

The same holds for any geometric braid subgroup B_Γ and its commensurator $C_\Gamma = Com_{B_m}(B_\Gamma)$ replacing B_n and C_n, respectively, in the above. Recall that C_Γ is just the stabiliser of Γ in $\pi_0(Diff(D^2, M))$. Because the groups C_Γ are conjugate for two curves Γ of the same type, and not even quasiconjugate otherwise, we also have the following.

Theorem 7.2. *The irreducible representations* λ_{B_m/C_Γ} *and* $\lambda_{B_m/C_{\Gamma'}}$ *of* B_m *are unitarily equivalent if and only if* Γ *and* Γ' *are curves of the same type, that is they enclose the same number of punctures.* □

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