Homotopy of knots and the Alexander polynomial

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ABSTRACT: Any knot in a 3-dimensional homology sphere is homotopic to a knot with trivial Alexander polynomial.

1. Introduction.

A knot K in a homology 3-sphere M has a well-defined symmetrized Alexander polynomial $\Delta_K(t)$ in the ring $\mathbf{Z}[t^{\pm 1}]$ of Laurent polynomials. This polynomial is an invariant of the ambient isotopy class of K in M. It is natural to ask whether the *homotopy* class $[K] \in \pi_1(M)$ imposes a constraint on the Alexander polynomial $\Delta_K(t)$. In this note, we will prove that there is no constraint.

Theorem 1. *If K is a knot in a homology sphere M^3, then K is homotopic to a knot $K_1 \subset M$ with trivial Alexander polynomial: $\Delta_{K_1}(t) = 1$.*

The homotopy may be assumed to fix a given basepoint. This is a consequence of a more general result regarding Seifert forms. As is well-known, any oriented knot K in an oriented homology sphere M^3 is the boundary of an oriented surface $F^2 \subset M$ whose genus we call g. Associated with F is the Seifert form V, which may be considered as a $2g \times 2g$ matrix with integer entries, once a basis for $H_1(F)$ is chosen.

Theorem 2. *Let K, M, F and V be as above and W any $2g \times 2g$ integral matrix such that $W - W^T = V - V^T$, where V^T is the transpose of V. Then K is homotopic in M to a knot with Seifert form W.*

According to Seifert [S], the classical knot polynomials are exactly those Laurent polynomials $p(t)$ with integer coefficients satisfying

$$p(1) = 1, \quad p(1/t) = p(t). \tag{1}$$

Corollary 3. *Within any homotopy class of knots in a homology sphere, the set of Alexander polynomials realized is exactly the set of integral Laurent polynomials satisfying (1).*

Corollary 3 can be seen to follow from Theorem 2, or directly from Theorem 1 by using the trick of adding a small local classical knot (by a homotopy). Freedman [F] proved that any homology 3-sphere M is the boundary of a contractible topological 4-manifold W, although there may not be such a smooth 4-manifold. In addition, Freedman-Quinn [FQ] showed that a knot in M with trivial Alexander polynomial is topologically slice, so we have the following application.

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Corollary 4. *Suppose that K is a knot in M, a homology 3-sphere which bounds the contractible topological 4-manifold W. Then K is homotopic in M to a knot which bounds a topologically locally-flat disk embedded in W.*

In the following sections we give an application to computation of certain generalized signatures, review basic definitions and, in the last section, prove Theorems 1 and 2. We would like to thank the referee for correcting a mistake in the first version of this paper.

2. Application to SU(2)-signatures.

The original motivation for considering this question comes from a skein theoretic presentation of an invariant for K in M. In [A], the invariant γ_K is introduced by counting $SU(2)$-representations of $\pi_1(M-K)$ with prescribed monodromy around a meridian of K, so that

$$\gamma_K : I_K \to \mathbf{Z}$$

is a continuous function from I_K, the unit circle in the complex plane minus the square roots of roots of $\Delta_K(t)$, into the integers. Two relations hold:

$$\begin{gathered} \gamma_{K_+}(\alpha) - \gamma_{K_-}(\alpha) = H(-\Delta_{K_+}(\alpha^2)\Delta_{K_-}(\alpha^2)) \\ \gamma_K(1) = 4\lambda(M) \end{gathered} \tag{2}$$

where $H(z)$ is the Heaviside function (that is, $H(z)=1$ if $z \geq 0$ and $H(z)=0$ if $z<0$), and $\lambda(M)$ is the Casson invariant of M. Here K_+ and K_- denote knots in M which are identical, except in a ball in which the oriented strings of the knot form a simple crossing which is positive or negative, respectively, using the usual conventions. The transition $K_- \leftrightarrow K_+$ is realized by a homotopy of the knot in M, and indeed knot homotopy is generated by such moves, together with ambient isotopy. Given a knot $K \subset M$, the knot K_1 provided by Theorem 1 can be used as the basis for an inductive calculation of γ_K.

Corollary 5. *For a knot K in a homology sphere M, the skein relations (2) uniquely determine the invariant γ_K.* □

3. Seifert surfaces and the Alexander polynomial.

Suppose that M is an oriented homology 3-sphere, that is, a compact 3-manifold with integral homology groups $H_*(M) \cong H_*(S^3)$. Let K a knot in M and let X denote the complement of a tubular neighborhood of K in M. Then by Alexander duality, $H_1(X) \cong \mathbf{Z}$. The kernel of the Hurewicz map $\pi_1(X) \to \mathbf{Z}$ defines a covering $\tilde{X} \to X$ whose deck transformation group $\mathrm{Aut}(\tilde{\mathrm{X}})$ is infinite cyclic. Denoting a generator of $\mathrm{Aut}(\tilde{\mathrm{X}})$ by t, we see that $H_1(\tilde{X})$ is a $\mathbf{Z}[t^{\pm}]$-module, whose order defines the Alexander polynomial $\Delta_K(t)$. That is, $\Delta_K(t)$ may be taken to be the determinant of a presentation matrix of $H_1(\tilde{X})$ as a $\mathbf{Z}[t^{\pm 1}]$-module. See, for example, [M] or [R] for further details.

A practical means of computing $\Delta_K(t)$ is provided by a Seifert surface. We first recall a construction (see [K]) for the reader's convenience. Consider a map $f : \partial X \to S^1$ such that the preimage of any point is the longitude of the knot K. Now remember that there are natural correspondences $H^1(X) \cong [X, S^1] \cong Z$ and $H^1(\partial X) \cong [\partial X, S^1] \cong \mathbf{Z} \oplus \mathbf{Z}$ and that we have the exact cohomology sequence

$$\cdots \to H^1(X) \to H^1(\partial X) \to H^2(X, \partial X) \to \cdots .$$

This implies that our map $f : \partial X \to S^1$ extends to a smooth map $\bar{f} : X \to S^1$. Then $\bar{f}^{-1}(p)$, for some regular value $p \in S^1$, is a 2-sided surface. Extending to the tubular neighborhood of K and discarding superfluous components, we obtain a connected orientable surface $F^2 \subset M$ with $K = \partial F$, called a Seifert surface for K.

There is a map $H_1(F) \to H_1(M - F)$ given by pushing a 1-cycle in F along the positive normal bundle. We denote the image of a 1-cycle α by α^+. The Seifert form associated to F is the bilinear form:

$$\begin{aligned} &V : H_1(F) \times H_1(F) \to \mathbf{Z} \\ &V(\alpha, \beta) = \mathrm{lk}(\alpha^+, \beta) \end{aligned}$$

where $\mathrm{lk}(\cdot, \cdot)$ denotes the linking number of two 1-cycles in M. Once a basis is chosen for $H_1(F)$, we may write V as a $2g \times 2g$ matrix with integer coefficients.

Two important properties of the Seifert form are well known. First, $V - V^T$ is the intersection form on $H_1(F)$ and is hence unimodular. This means that in a canonical (symplectic) basis for $H_1(F)$,

$$V - V^T = \begin{bmatrix} 0 & 1 & & & & & \\ -1 & 0 & & & & & \\ & & 0 & 1 & & & \\ & & -1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & 1 \\ & & & & & -1 & 0 \end{bmatrix} \equiv \oplus_g \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

This shows that the Seifert form is completely determined by its entries on and above the diagonal.

Secondly, $tV - V^T$, or equivalently, $t^{1/2}V - t^{-1/2}V^T$, is a presentation matrix for $H_1(\tilde{X})$ as a $\mathbf{Z}[t^{\pm 1}]$-module, and so

$$\Delta_K(t) = \det(t^{1/2}V - t^{-1/2}V^T).$$

Example 1: In S^3, the surface below has genus g and unknotted boundary.

Figure 1. A Seifert surface for the unknot.

In fact, if we choose the natural symplectic basis for $H_1(F)$, for this example

$$V = \oplus_g \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$$

so, of course, $\Delta_K(t) = 1$.

Example 2: Suppose that K is a knot in S^3 and M the result of $1/n$ surgery on K; that is, $M = S^3 - N(K) \cup_h (S^1 \times D^2)$ where $h : S^1 \times S^1 \to \partial N(K)$. If we let $L = S^1 \times \{0\} \subset M$, then $\Delta_L = \Delta_K$. Although it is by no means obvious, we will see that we can change L by a homotopy within M so that the Alexander polynomial of the new knot is trivial.

3: Proof of Theorems 1 and 2.

As before, let K be a knot in a homology 3-sphere M and choose a basepoint $*$ in K. If $F \subset M$ is a Seifert surface for K, then F has a 1-dimensional spine consisting of a bouquet of disjoint simple loops based at $*$. A collapse of F to a neighborhood of the spine may be covered by an ambient isotopy of M. This means that F is ambient isotopic (with $*$ fixed) to a small disk with $2g$ ribbons, which we denote by $R_1, \ldots, R_{2g}$, attached to the boundary.

We now describe a move on the ribbons. Suppose that parts of R_i and R_j are contained in some 3-ball in M as shown in the left half of Figure 3.1. Consider the move that changes the embedding to be as in the right side of the figure. This local move may change the embedded surface F and the knot K and is generally not realizable by an ambient isotopy. It is, however, realized by a homotopy of K in M, and in fact a regular homotopy of F. This move was used by Kauffman [Ka], where it is called a *pass equivalence*; he noted that the Arf invariant of a knot is unchanged by this move.

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Figure 2: A move on ribbons.

This pass equivalence move changes the Seifert form associated with F in a simple way. We will let $r_i \in H_1(F)$ be the cycle defined by R_i. We can assume that the collection $\{r_i\}$ forms a canonical basis for $H_1(F)$. If we let V be the Seifert form before the move and V' the Seifert form after, then we have, if $i \neq j$

$$V'(r_i, r_j) = V(r_i, r_j) \pm 1, \quad V'(r_j, r_i) = V(r_j, r_i) \pm 1$$

while all other entries in the Seifert form are unchanged.

If $i = j$ the corresponding entry is changed by ± 2. However, a different move, introducing a simple twist in the ribbon R_i has the effect of changing $V(r_1, r_1)$ by ± 1, and can also be accomplished by a knot homotopy.

Earlier we noted that in a canonical basis, the condition

$$V - V^T = \oplus_g \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

implies that the entries on or above the diagonal determine the Seifert form. The moves described above have the effect of changing exactly one entry on or above the diagonal, by ± 1. This means that given W, a $2g \times 2g$ matrix with integer entries satisfying

$$W - W^T = \oplus_g \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

we may find a sequence of these moves to modify K and F by a homotopy so that the Seifert form becomes W, and Theorem 2 follows. In particular, we may realize the form of Example 1, and Theorem 1 is proved. $\square$

Remark on genus: One should note that, although the Seifert form may be modified radically by a homotopy, the size $2g \times 2g$ of the form will not, in general, be reduced. Indeed, within the homotopy class of K in M, a lower bound for the genus g is the algebraic genus. For any element k in the commutator subgroup of a group G, we can define its *algebraic genus* to be the minimal n for which there is an expression $k = \prod_{i=1}^n x_i y_i x_i^{-1} y_i^{-1}, \ \ x_i, y_i \in G$. It is an easy observation that if K bounds a surface of genus g in M, then its homotopy class $[K]$ has an expression as a product of g commutators in $\pi_1(M)$.

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