New developments in the theory of Artin's braid groups

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Dedicated to the memory of Jim van Buskirk.

1 Introduction.

The braid groups B_n were introduced by E. Artin three-quarters of a century ago [1]. Despite their ripe old age, and an enormous amount of attention paid to them by mathematicians and physicists, these groups still provide us with big surprises. Indeed, the major subject of the present conference – quantum invariants of 3-manifolds – was essentially born when V. Jones discovered, about fifteen years ago, new representations of the braid groups. In this informal discussion, I will touch on that discovery, and more recent ones, regarding this wonderful family of groups, and raise a few questions about them which are still open. Along the way, I will also discuss some known results which deserve to be better known – some do not seem to be mentioned in the existing literature. However, this talk is not, by any means, a complete survey.

2 The many faces of braid groups.

One of the interesting things about the braid groups B_n is they can be defined in so many ways, each providing a unique insight. I will briefly describe some of these definitions, punctuated by facts about B_n which are revealed from the various points of view. You can check the excellent references [1], [2], [6], [9], [12], [17] for more information.

Definition 1: Braids as particle dances. Look at n particles located at distinct points in a plane. To be definite, suppose they begin at the integer points $\{1, \ldots, n\}$ in the complex plane \mathbb{C} . Now let them move around in trajectories

$$\beta(t) = (\beta_1(t), \dots, \beta_n(t)), \quad \beta_i(t) \in \mathbb{C}$$

A braid is then such a time history $\beta = \beta(t), 0 \le t \le 1$ with the proviso that the particles are noncolliding:

$$\beta_i(t) \neq \beta_i(t)$$
 if $i \neq j$

and end at the spots they began, but possibly permuted:

$$\beta_i(0) = i, \quad \beta_i(1) \in \{1, \dots, n\}.$$

If one braid can be deformed continuously into another (through the class of braids), the two are considered equivalent – we may as well say equal.

Braids α and β can be multiplied: one dance following the other, at double speed. The product is associative but not commutative (in general). The identity is to stand still, and

each dance has an inverse; doing the dance in reverse. These (deformation classes of) dances form the group B_n .

A braid β defines a permutation $i \to \beta_i(1)$ which is a well-defined element of the permutation group Σ_n . This is a homomorphism with kernel, by definition, the subgroup $P_n < B_n$ of pure braids. P_n is sometimes called the *colored* braid group, as the particles can be regarded as having identities, or colors. P_n is of course normal in B_n , of index n!, and there is an exact sequence

$$1 \to P_n \to B_n \to \Sigma_n \to 1$$
.

Definition 2: Braids as strings in 3-D. This is the usual and visually appealing picture. A braid can be viewed as its graph, or timeline, in real x, y, t-space, monotone in the t direction. The complex part is described as usual by $x + y\sqrt{-1}$. The product is then a concatenation of braided strings.

This viewpoint provides the connection with knots. A braid β defines a knot or link $\hat{\beta}$, its closure, by connecting the endpoints in a standard way.

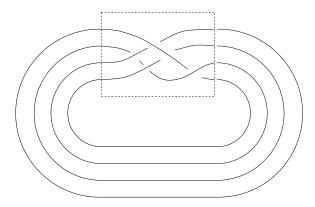


Figure 1: The closure of a braid.

Definition 3: B_n as a fundamental group. In complex n-space \mathbb{C}^n consider the big diagonal

$$\Delta = \{(z_1, \dots, z_n); \quad z_i = z_j, \text{ some } i < j\} \subset \mathbb{C}^n.$$

Using the basepoint (1, 2, ..., n), we see that

$$P_n = \pi_1(\mathbb{C}^n \setminus \Delta).$$

In other words, pure braid groups are fundamental groups of complements of a special sort of complex *hyperplane arrangement*, itself a deep and complicated subject.

To get the full braid group we need to take the fundamental group of the *configuration* space, of orbits of the obvious action of Σ_n upon $\mathbb{C}^n \setminus \Delta$. Thus

$$B_n = \pi_1((\mathbb{C}^n \setminus \Delta)/\Sigma_n).$$

Notice that since the singularities have been removed, the projection

$$\mathbb{C}^n \setminus \Delta \longrightarrow (\mathbb{C}^n \setminus \Delta)/\Sigma_n$$

is actually a covering map. As is well-known, covering maps induce injective homomorphisms at the π_1 level, so this is another way to think of the inclusion $P_n \subset B_n$.

It was observed in [14] that $\mathbb{C}^n \setminus \Delta$ has trivial homotopy groups in dimension greater than one. That is, it is an Eilenberg-Maclane space, also known as a $K(P_n, 1)$. Therefore its cohomology groups coincide with the group cohomology of P_n . By covering theory, the quotient space $(\mathbb{C}^n \setminus \Delta)/\Sigma_n$ also has trivial higher homotopy, so it is a $K(B_n, 1)$. Since these spaces have real dimension 2n, this view of braid groups gives us the following observation.

Theorem 2.1 The groups B_n and P_n have finite cohomological dimension.

If a group contains an element of finite order, standard homological algebra implies that the cohomological dimension of the group must be infinite. Thus there are no braids of finite order.

Corollary 2.2 The braid groups are torsion-free.

Finally, we note that the space $(\mathbb{C}^n \setminus \Delta)/\Sigma_n$ can be identified with the space of all complex polynomials of degree n which are monic and have n distinct roots

$$p(z) = (z - r_1) \cdots (z - r_n).$$

This is one way in which the braid groups play a role in classical algebraic geometry, as fundamental groups of such spaces of polynomials.

Definition 4: The algebraic braid group. B_n can be regarded algebraically as the group presented with generators $\sigma_1, \ldots, \sigma_{n-1}$, where σ_i is the braid with one crossing, with the string at level i (counting from the bottom) crossing over the one at level i+1 and the other strings going straight across. In Figure 1, for example, the braid could be written either as $\sigma_2^{-1}\sigma_1^{-1}\sigma_3\sigma_2\sigma_1^{-1}$ or $\sigma_2^{-1}\sigma_3\sigma_1^{-1}\sigma_2\sigma_1^{-1}$ (or indeed many other ways).

These generators are subject to the relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i, |i - j| > 1,$$

$$\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j, |i - j| = 1.$$

We can take a whole countable set of generators $\sigma_1, \sigma_2, \ldots$ subject to the above relations, which defines the infinite braid group B_{∞} . If we consider the (non-normal) subgroup generated by $\sigma_1, \ldots, \sigma_{n-1}$, these algebraically define B_n . Notice that this convention gives "natural" inclusions $B_n \subset B_{n+1}$ and $P_n \subset P_{n+1}$.

Going the other way, if one forgets the last string of an n+1-braid the result is an n-braid. But strictly speaking, this is only a well defined homomorphism for pure braids, (or at best for the subgroup of braids in which the string beginning at the point n+1 also ends there). Later, we will have a use for this forgetful map

$$f: P_{n+1} \to P_n$$

It is easy to see that f is a left inverse of the inclusion, or in other words a retraction in the category of groups.

Artin Combing. We now have the ingredients for the combing technique, by which Artin solved the word problem for pure braid groups, and therefore for the full braid groups.

Let β be a pure *n*-braid and $f(\beta)$ the pure n-1 braid obtained by forgetting the last string, but then by inclusion, regard $f(\beta)$ in P_n . Then β and $f(\beta)$ can be visualized as the same braid, except the last string has been changed so as to have no interaction with the other strings. Let K be the kernel of f. Then it is easy to verify that $f(\beta)^{-1}\beta \in K$ and the map

$$\beta \to (f(\beta), f(\beta)^{-1}\beta)$$

maps P_n bijectively onto the cartesian product $P_{n-1} \times K$. However, the multiplicative structure is that of a *semidirect* product, as happens whenever we have a group retraction. Also notice that every element of K can be represented by a braid in which the first n-1 strands go straight across. In this way we identify K with the fundamental group of the complement of the points $\{1, \ldots, n-1\}$ in the plane, which is a free group: $K \cong F_{n-1}$. This process can then be iterated on P_{n-1} to obtain the Artin normal form: P_n is an iterated semidirect product of free groups $F_1, F_2, \ldots, F_{n-1}$.

Theorem 2.3 There is an algorithmic solution to the word problem in P_n and B_n .

Now to solve the word problem in P_n to decide whether a braid β expressed as a word in the generators is trivial in the group – construct the unique expansion

$$\beta = \beta_1 \beta_2 \cdots \beta_{n-1}$$

from the Artin combing, with $\beta_j \in F_j$. The word problem in free groups being very easy (two words represent the same group element iff their reduced forms are equal) we have solved the word problem for P_n . Since P_n has finite index in B_n , it is straightforward to extend this to a solution of the word problem in the full braid group B_n . For later reference, we will call the vector

$$(\beta_1,\beta_2,\cdots,\beta_{n-1})$$

the "Artin coordinates" of β .

Definition 5: B_n as a mapping class group. Going back to the first definition, imagine the particles are in a sort of planar jello and pull their surroundings with them as they dance about. Topologically speaking, the motion of the particles extends to a continuous family of homeomorphisms of the plane (or of a disk, fixed on the boundary). This describes an equivalence between B_n and the mapping class of D_n , the disk D with n punctures (marked points). That is, B_n can be considered as the group of homeomorphisms of D_n fixing ∂D and permuting the punctures, modulo isotopy fixing $\partial D \cup \{1, \ldots, n\}$.

Definition 6: B_n as a group of automorphisms. A mapping class [h], where $h: D_n \to D_n$ gives rise to an automorphism $h_*: F_n \to F_n$ of free groups. Using the interpretation of braids as mapping classes, this defines a homomorphism

$$B_n \to Aut(F_n),$$

which Artin showed to be faithful, i. e. injective.

The generator σ_i acts as

$$x_{i} \to x_{i}x_{i+1}x_{i}^{-1}$$

$$x_{i+1} \to x_{i}$$

$$x_{j} \to x_{j}, j \neq i, i+1.$$
(1)

Theorem 2.4 (Artin) Under the identification described above, B_n is the set of automorphisms $h \in Aut(F_n)$ of the form

$$h(x_i) = w_j^{-1} x_j w_j,$$

where w_j are words in F_n , and satisfying $h(x_1 \cdots x_n) = x_1 \cdots x_n$.

This point of view gives further insight into the group-theoretic properties of braid groups. Recall that a group G is residually finite if for every $g \in G$ there is a homomorphism $h: G \to F$ onto a finite group F such that h(g) is not the identity. It is well-known that free groups are residually finite. A nice argument due to Baumslag [4] (see [21]) shows that the group of automorphisms of a finitely generated residually finite group is itself residually finite. Thus $Aut(F_n)$ is residually finite, as are all its subgroups.

Theorem 2.5 B_n is residually finite.

It follows immediately (see [21]) that

Theorem 2.6 B_n is Hopfian, that is, not isomorphic with a proper quotient of itself.

We close this section with a theorem of W. Thurston. An infinite group is called *automatic* if is well-modelled (in a well-defined technical sense which I won't elaborate here) by a finite-state automaton. The standard reference is [11].

Theorem 2.7 (Thurston) B_n is automatic.

This implies, for example, that the word problem can be solved by an algorithm which is quadratic in the length of the input.

3 Representations of B_n

This is a very big subject, which I will just touch upon. By a representation of a group we will mean a homomorphism of the group into a group of matrices, or more generally into some other group, or ring or algebra. Often, but not always, we want the target to be finite-dimensional. We've already encountered the Artin representation $B_n \to Aut(F_n)$, which is faithful. Here the target group is far from being "finite-dimensional."

One of the classical representations of the braid groups is the Burau representation, which can be described as follows. Consider the definition of B_n as the mapping class group of the punctured disk D_n (Definition 5). As already noted, the fundamental group of D_n is a free group, with generator x_i represented by a loop, based at a point on the boundary of the disk, which goes once around the i^{th} puncture. Consider the subgroup of $\pi_1(D_n)$ consisting of all words in the x_i whose exponent sum is zero. This is a normal subgroup, and so defines a regular covering space $\bar{D}_n \to D_n$. The group of covering translations is infinite cyclic. Therefore, the homology $H_1(\bar{D}_n)$ can be considered as a module over the polynomial ring $Z[t,t^{-1}]$, where t represents the generator of the covering translation group. A braid β can be represented as (an isotopy class of) a homeomorphism $\beta:D_n\to D_n$ fixing the basepoint. This lifts to a homeomorphism $\bar{\beta}:\bar{D}_n\to\bar{D}_n$, which is unique if we insist that it fix some particular lift of the basepoint. The induced homomorphism on homology, $\bar{\beta}_*:H_1(\bar{D}_n)\to H_1(\bar{D}_n)$ is a linear map of these finite-dimensional modules, and so can be represented by a matrix with entries in $\mathbb{Z}[t,t^{-1}]$. The mapping

$$\beta \to \bar{\beta}_*$$

is the Burau representation of B_n . An explicit matrix form, and its relations with the Alexander polynomial can be found in [6].

It has been known for many years that this representation is faithful for $n \leq 3$, and it is only within the last decade that it was found to be unfaithful for any n at all. John Moody showed in 1993 [23] that it is unfaithful for $n \geq 9$. This has since been improved by Long and Paton [20] and very recently by Bigelow to $n \geq 5$. The case n = 4 remains open, as far as I am aware.

Another very important representation is the one defined by Jones [16] which gave rise to his famous knot polynomial, and the subsequent revolution in knot theory. The version I will discuss is more thoroughly described in [17]; it is based on the Kauffman bracket, an elementary combinatorial approach to the Jones polynomial. First we need to describe the Temperley-Lieb Algebras \mathcal{T}_n , in their geometric form. The elements of \mathcal{T}_n are something like braids: we consider strings in a box, visualized as a square in the plane, endpoints being exactly n specified points on each of the left and right sides. The strings are not required to be monotone, or even to run across from one side to the other. There also may be closed components. Really what we are looking at are "tangle" diagrams. Two tangle diagrams are considered equal if there is a planar isotopy, fixed on the boundary of the square, taking one to the other.

Now we let A be a fixed complex number (regarded as a parameter), and formally define \mathcal{T}_n to be the complex vector space with basis the set of all tangles, as described above,

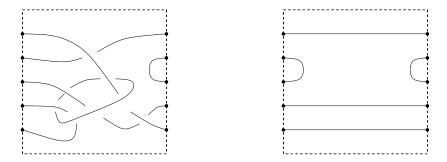


Figure 2: A typical element in \mathcal{T}_5 and the generator e_3 .

but modulo the following relations, which correspond to similar relations used to define Kauffman's bracket version of the Jones polynomial.

$$K \cup \bigcirc = dK$$
, $d = -A^2 - A^{-2}$

Figure 3: Relations in \mathcal{T}_n .

The first relation means that we can replace a tangle with a crossing by a linear combination of two tangles with that crossing removed in two ways. As usual, the pictures mean that the tangles are identical outside the part pictured. The second relation means that we can remove any closed curve in the diagram, if it does not have any crossings with the rest of the tangle, at the cost of multiplying the tangle by the scalar $-A^2 - A^{-2}$. Using the relations, we see that any element of \mathcal{T}_n can be expressed as a linear combination of tangles which have no crossings and no closed curves - that is, disjoint planar arcs connecting the 2n points of the boundary. This gives a finite generating set, which (for generic values of A) can be shown to be a basis for \mathcal{T}_n as a vector space. But there is also a multiplication of \mathcal{T}_n , a concatenation of tangles, in exactly the same way braids are multiplied. This enables us to consider \mathcal{T}_n to be generated as an algebra by the elements e_1, \ldots, e_{n-1} . In e_i all the strings go straight across, except those at level i and i+1 which are connected by short caps; the generator e_3 of \mathcal{T}_5 is illustrated in Figure 2. The identity of this algebra is simply the diagram consisting of n horizontal lines (just like the identity braid). \mathcal{T}_n can be described abstractly as the associative algebra with the generators e_1, \ldots, e_{n-1} , subject to the relations:

$$e_i e_{i\pm 1} e_i = e_i$$
 $e_i^2 = (-A^2 - A^{-2})e_i$

It is an enjoyable exercise to verify these relations from the pictures. Now the Jones representation $J: B_n \to \mathcal{T}_n$ can be described simply by considering a braid diagram as an element of the algebra. In terms of generators, this is just

$$J(\sigma_i) = A + A^{-1}e_i$$

I will have more to say about representations of B_n later.

4 Ordering braid groups

This is one of the most exciting of the recent developments in braid theory. Call a group G right orderable if its elements can be given a strict total ordering < which is right-invariant:

$$\forall x,y,z \in G, \quad x < y \Rightarrow xz < yz.$$

Theorem 4.1 (Dehornoy[10]) B_n is right-orderable.

Interestingly, I know of three quite different proofs. The first is Dehornoy's, the second is one that was discovered jointly by myself and four other topologists. We were trying to understand difficult technical aspects of Dehornoy's argument, then came up with quite a different way of looking at exactly the same ordering, but using the view of B_n as the mapping class group of the punctured disk D_n . Yet a third way is due to Thurston, using the fact that the universal cover of D_n embeds in the hyperbolic plane. Here are further details.

Dehornoy's approach: It is routine to verify that a group G is right-orderable if and only if there exists a subset Π (positive cone) of G satisfying:

- (1) $\Pi \cdot \Pi = \Pi$
- (2) The identity element does not belong to Π , and for every $g \neq 1$ in G exactly one of $g \in \Pi$ or $g^{-1} \in \Pi$ holds.

One defines the ordering by g < h iff $hg^{-1} \in \Pi$.

Dehornoy's idea is to call a braid *i*-positive if it is expressible as a word in $\sigma_j, j \geq i$ in such a manner that all the exponents of σ_i are positive. Then define the set $\Pi \subset B_n$ to be all braids which are *i*-positive for some $i = 1, \ldots, n-1$. To prove (1) above is quite easy, but (2) requires an extremely tricky argument.

Here is the point of view advocated in [13]. Consider B_n as acting on the complex plane, as described above. Our idea is to consider the image of the real axis $\beta(\mathbb{R})$, under a mapping class $\beta \in B_n$. Of course there are choices here, but there is a unique "canonical form" in which (roughly speaking) $\mathbb{R} \cap \beta(\mathbb{R})$ has the fewest number of components. Now declare a braid β to be positive if (going from left to right) the first departure of the canonical curve $\beta(\mathbb{R})$ from \mathbb{R} itself is into the *upper* half of the complex plane. Amazingly, this simple idea works, and gives exactly the same ordering as Dehornoy's combinatorial definition.

Finally, Thurston's idea for ordering B_n again uses the mapping class point of view, but a different way at looking at ordering a group. This approach, which has the advantage of defining infinitely many right-orderings of B_n is described by H. Short and B. Wiest in [27].

The Dehornoy ordering (which is discrete) occurs as one of these right-orderings – others constructed in this way are order-dense. A group G acts on a set X (on the right) if the mapping $x \to xg$ satisfies: x(gh) = (xg)h and x1 = x. An action is *effective* if the only element of G which acts as the identity is the identity $1 \in G$. The following is a useful criterion for right-orderability:

Lemma 4.2 If the group G acts effectively on \mathbb{R} by order-preserving homeomorphisms, then G is right-orderable.

By way of a proof, consider a well-ordering of the real numbers. Define, for g and $h \in G$,

$$g < h \Leftrightarrow xg < xh$$
 at the first $x \in \mathbb{R}$ such that $xg \neq xh$.

It is routine to verify that this defines a right-invariant strict total ordering of G. (By the way, we could have used any ordered set in place of \mathbb{R} .) For those wishing to avoid the axiom of choice (well-ordering \mathbb{R}) we could have just used an ordering of the rational numbers.

The universal cover \tilde{D}_n of D_n can be embedded in the hyperbolic plane \mathbb{H}^2 in such a way that the covering translations are isometries. This gives a hyperbolic structure on D_n . It also gives a beautiful tiling of \mathbb{H}^2 , illustrated in Figure 4 for the case n=2.

Choose a basepoint $* \in \partial D$ and a specific lift $\tilde{*} \in \mathbb{H}^2$. Now a braid is represented by a homeomorphism of D_n , which fixes *. This homeomorphism lifts to a homeomorphism of $\tilde{D_n}$, unique if we specify that it fixes $\tilde{*}$. In turn, this homeomorphism extends to a homeomorphism of the boundary of $\tilde{D_n}$ But in fact, this homeomorphism fixes the interval of $\partial \tilde{D_n}$ containing $\tilde{*}$, and if we identify the complement of this interval with the real line \mathbb{R} , a braid defines a homeomorphism of \mathbb{R} . This defines an action of B_n upon \mathbb{R} by order-preserving homeomorphisms, and hence a right-invariant ordering of B_n .

Two-sided invariance? Any right-invariant ordering of a group can be converted to a left-invariant ordering, by comparing inverses of elements, but that ordering is in general different from the given one. We will say that a group G with strict total ordering < is fully-ordered, or bi-ordered, if

$$x < y \Rightarrow xz < yz$$
 and $zx < zy, \forall x, y, z \in G$.

There are groups which are right-orderable but not bi-orderable – in fact the braid groups!

Proposition 4.3 (N. Smythe) For n > 2 the braid group B_n cannot be fully ordered.

The reason for this is that there exists a nontrivial element which is conjugate to its inverse: take $x = \sigma_1 \sigma_2^{-1}$ and $y = \sigma_1 \sigma_2 \sigma_1$ and note that $yxy^{-1} = x^{-1}$. In a bi-ordered group, if 1 < x then $1 < yxy^{-1} = x^{-1}$, contradicting the other conclusion $x^{-1} < 1$. If x < 1 a similar contradiction arises.

Theorem 4.4 The pure braid groups P_n can be bi-ordered.

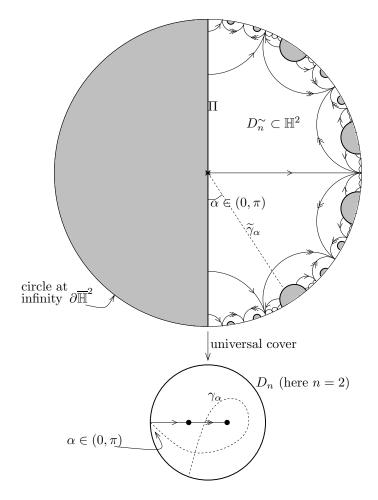


Figure 4: The universal cover of a twice-punctured disk, with a lifted geodesic. (Courtesy of H. Short and B. Wiest [27]

This theorem was first noticed by J. Zhu, and the argument appears in [25], based on the result of Falk and Randall [15] that the pure braid groups are "residually torsion-free nilpotent." Later, in joint work with Djun Kim, we discovered a really natural, and I think beautiful, way to define a bi-invariant ordering of P_n . We've already done half the work, by discussing Artin combing. Now we need to discuss ordering of free groups.

Lemma 4.5 For each $n \geq 1$, the free group F_n has a bi-invariant ordering < with the further property that it is invariant under any automorphism $\phi : F_n \to F_n$ which induces the identity upon abelianization: $\phi_{ab} = id : \mathbb{Z}^n \to \mathbb{Z}^n$.

The construction depends on the Magnus expansion of free groups into rings of formal power series. Let F be a free group with free generators x_1, \ldots, x_n . Let $Z[[X_1, \ldots, X_n]]$ denote the ring of formal power series in the *non-commuting* variables X_1, \ldots, X_n . Each term in a formal power series has a well-defined (total) degree, and we use O(d) to denote terms of degree $\geq d$. The subset $\{1 + O(1)\}$ is actually a multiplicative subgroup of $Z[[X_1, \ldots, X_n]]$.

Moreover, there is an embedding

$$\mu: F \to Z[[X_1, \ldots, X_n]]$$

defined by

$$\mu(x_i) = 1 + X_i$$

$$\mu(x_i^{-1}) = 1 - X_i + X_i^2 - X_i^3 + \cdots$$
(2)

There is a very nice proof that μ is injective in [21], as well as discussion of some if its properties. One such property is that commutators have zero linear terms. For example (dropping the μ)

$$[x_1, x_2] = x_1 x_2 x_1^{-1} x_2^{-1}$$

$$= (1 + X_1)(1 + X_2)(1 - X_1 + X_1^2 - \dots)(1 - X_2 + X_2^2 - \dots)$$

$$= 1 + X_1 X_2 - X_2 X_1 + O(3).$$
(3)

Now there is a fairly obvious ordering of $Z[[X_1, \ldots, X_n]]$. Write a power series in ascending degree, and within each degree list the monomials lexicographically according to subscripts). Given two series, order them according to the coefficient of the first term (when written in the standard form just described) at which they differ. Thus, for example, 1 and $[x_1, x_2]$ first differ at the X_1X_2 term, and we see that $1 < [x_1, x_2]$. It is not difficult to verify that this ordering, restricted to the group $\{1 + O(1)\}$ is invariant under both left- and right-multiplication.

Now we define an ordering of P_n . If α and β are pure braids, compare their Artin coordinates (as described earlier)

$$(\alpha_1, \alpha_2, \dots, \alpha_{n-1})$$
 and $(\beta_1, \beta_2, \dots, \beta_{n-1})$

lexicographically, using within each F_k the Magnus ordering described above. This all needs choices of conventions, for example, for generators of the free groups, described in detail in [18]. The crucial fact is that the action associated with the semidirect product, by automorphisms φ , has the property mentioned in Lemma 4.5. We recall the definition of a positive braid according to Garside: a braid is Garside-positive if it can be expressed as a word in the standard generators σ_i with only positive exponents.

Theorem 4.6 (Kim-Rolfsen) P_n has a bi-ordering with the property that Garside-positive pure braids are greater than the identity, and the set of all Garside-positive pure braids is well-ordered by the ordering.

Algebraic consequences: The orderability of the braid groups has implications beyond what we already knew - e. g. that they are torsion-free. In the theory of representations

of a group G, it is important to understand the group algebra $\mathbb{C}G$ and the group ring $\mathbb{Z}G$. These rings also play a role in the theory of Vassiliev invariants. A basic property of a ring would be whether it has (nontrivial) zero divisors. I don't know who first discovered the following. It appears in [26], which gives a very good account of orderings and group rings.

Proposition 4.7 If R is a ring without zero divisors, and G is a right-orderable group, then the group ring RG has no zero divisors. Moreover, the only units of RG are the monomials $rg, g \in G$, r a unit of R.

Proposition 4.8 (Malcev, Neumann) If G has a bi-invariant ordering, then its group ring $\mathbb{Z}G$ embeds in a division algebra, that is, an extension in which all nonzero elements have inverses.

These results give us new information about the group rings of the braid groups.

Theorem 4.9 $\mathbb{Z}B_n$ has no zero divisors. Moreover, $\mathbb{Z}P_n$ embeds in a division algebra.

A proof of the theorem of Malcev and Neumann [22] can be found in [24]. I do not know if $\mathbb{Z}B_n$ has an embedding in a division algebra. I think this is a good open question. Another good open question is, which subgroups of $Aut(F_n)$ are right-orderable? Of course, $Aut(F_n)$ itself is not right-orderable, because it has elements of finite order, e.g. permuting the generators.

A final note regarding orderings: As we've seen, the methods we've used for ordering B_n and P_n are quite different. One might hope there could be compatible orderings: a bi-ordering of P_n which extends to a right-invariant ordering of B_n . But, in recent work with Akbar Rhemtulla, we showed this is hopeless!

Theorem 4.10 (Rhemtulla, Rolfsen). For $n \geq 5$, there is no right-invariant ordering of B_n , which, upon restriction to P_n , is also left-invariant.

5 Linearity of the braid groups.

This is a news flash, having been announced after the conference, which is the occasion of this report, has finished. However, it is certainly one of the most exciting of the recent developments of braid theory. It has long been questioned, whether the braid groups are linear, meaning that there is a faithful representation $B_n \to GL(V)$ for some finite dimensional vector space V. A candidate had been the so-called Burau representation, but as already mentioned it has been known for several years ([23], [20]) that Burau is unfaithful, in general. Just recently, Bachmuth [3] published an argument that the Gassner representation (of pure braid groups) is faithful, from which it would follow that the full braid groups are also linear. Unfortunately, his arguments had serious gaps, which were pointed out in the review [7].

It now seems the question has been settled after all, according to an announcement of Stephen Bigelow. An alternative proof has been announced by Krammer, using the same representation, but different methods. They use a representation defined very much like the Burau representation. But instead of a covering of the punctured disk D_n , they use a covering of the configuration space of pairs of points of D_n , upon which B_n also acts. This action induces a linear representation in the homology of an appropriate covering, and provides just enough extra information to give a faithful representation!

Theorem 5.1 (Bigelow, Krammer) The braid groups are linear.

In fact, Bigelow has announced that the BMW representation (Birman, Murakami, Wenzl) [8] is also faithful. Another open question is whether the Jones representation $J: B_n \to T_n$, discussed earlier, is faithful. Is there any way to use orderability of B_n to study its representations? For example, one might try to order T_n and T_n in such a way that the Jones representation is order-preserving. If one could show that $\beta < \gamma$ implies $T_n = T_n$, this would establish that $T_n = T_n$ is faithful, and give another proof of the linearity of $T_n = T_n$.

Linearity of B_n is the latest in the seemingly boundless store of secrets of the marvelous braid groups of Artin. I believe they will continue to fascinate and surprise us for years to come. (Added in proof: these results appear in [19] and [5]).

References

- [1] E. Artin, Theorie der Zöpfe, Abh. Math. Sem. Hamburg. Univ. 4(1926), 47 72.
- [2] E. Artin, Theory of braids, Annals of Math. (2) 48(1947), 101 126.
- [3] S. Bachmuth, Braid groups are linear groups, Adv. Math. 121(1996), 50 61.
- [4] G. Baumslag, Automorphism groups of residually finite groups, J. London Math. Soc. 38(1963), 117–118.
- [5] Bigelow, Stephen J. Braid groups are linear. J. Amer. Math. Soc. 14 (2001), no. 2, 471–486 (electronic).
- [6] J. Birman, Braids, links and mapping class groups, Annals of Math. Studies 82, Princeton Univ. Press, 1974.
- [7] J. Birman, review of [Ba], Math. Reviews 98h:20061.
- [8] J. Birman and H. Wenzl, Braids, link polynomials and a new algebra, *Trans. Amer. Math. Soc.* 313(1989), 249 273.
- [9] G. Burde and H. Zieschang, *Knots*, de Gruyter, 1985.
- [10] P. Dehornoy, Braid groups and left distributive operations, *Trans Amer. Math. Soc.* 345(1994), 115 150.

- [11] D. Epstein, J. Cannon, D. Holt, S. Levy, M. Paterson, W. Thurston, Word processing in groups, Jones and Bartlett, 1992.
- [12] R. Fenn, An elementary introduction to the theory of braids, notes by B. Gemein, 1999, available at the author's website.
- [13] R. Fenn, M. Green, D. Rolfsen, C. Rourke, B. Wiest, Ordering the braid groups, Pacific J. Math. 191(1999), 49–74.
- [14] E. Fadell and L. Neuwirth, Configuration spaces, Math. Scand. 10(1962), 111 118.
- [15] M. Falk and R. Randell, Pure braid groups and products of free groups, *Braids (Santa Cruz 1986)*, *Contemp. Math.* 78 217–228.
- [16] V. F. R. Jones, A polynomial invariant for knots via Von Neumann algebras, Bull. Amer. Math. Soc. 12(1985), 103–111.
- [17] L. Kauffman, Knots and Physics, World Scientific, 1993.
- [18] D. Kim and D. Rolfsen, An ordering for groups of pure braids and hyperplane arrangements, to appear, Can. Math. Bulletin.
- [19] D. Krammer, The braid group B_4 is linear. Invent. Math. 142 (2000), no. 3, 451–486.
- [20] D. Long and M. Paton, The Burau representation is not faithful for $n \ge 6$, Topology 32(1993), 439 447.
- [21] W. Magnus, A. Karrass and D. Solitar, Combinatorial group theory, Wiley 1966.
- [22] A. I. Mal'cev, On the embedding of group algebras in division algebras, Dokl. Akad. Nauk SSSR 60 (1948) 1944-1501.
- [23] J. Moody, The faithfulness question for the Burau representation, *Proc. Amer. Math. Soc.* 119(1993), 439 447.
- [24] R. Mura and A. Rhemtulla, *Orderable groups*, Lecture Notes in Pure and Applied Mathematics, vol. 27, Marcel Dekker, New York, 1977.
- [25] D. Rolfsen, J. Zhu, Braids, orderings and zero divisors, J. Knot Theory and its Ramifications 7(1998), 837-841.
- [26] D. S. Passman, The algebraic structure of group rings, Pure and Applied Mathematics, Wiley Interscience, 1977.
- [27] H. Short and B. Wiest, Orderings of mapping class groups after Thurston, to appear, l'Enseignement Math.