Orderable 3-manifold groups

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Abstract

We investigate the orderability properties of fundamental groups of 3-dimensional manifolds. Many 3-manifold groups support left-invariant orderings, including all compact $\mathbb{P}^2$-irreducible manifolds with positive first Betti number. For seven of the eight geometries (excluding hyperbolic) we are able to characterize which manifolds' groups support a left-invariant or bi-invariant ordering. We also show that manifolds modelled on these geometries have virtually bi-orderable groups. The question of virtual orderability of 3-manifold groups in general, and even hyperbolic manifolds, remains open, and is closely related to conjectures of Waldhausen and others.

1 Introduction

A group G is called *left-orderable* (LO) if its elements can be given a (strict) total ordering $<$ which is left invariant, meaning that $g < h \Rightarrow fg < fh$ if $f, g, h \in G$. We will say that G is *bi-orderable* (O) if it admits a total ordering which is simultaneously left and right invariant (historically, this has been called "orderable"). A group is called *virtually left-orderable* or *virtually bi-orderable* if it has a finite index subgroup with the appropriate property.

It has recently been realized that many of the groups which arise naturally in topology are left-orderable. Dehornoy provided a left-ordering for the Artin braid groups [De]; see also [FGRRW] and [SW]. Rourke and Wiest [RW] extended this, showing that mapping class groups of all Riemann surfaces with nonempty boundary (and possibly with punctures) are left-orderable. In general these groups are not bi-orderable. On the other hand, the pure Artin braid groups are known to be bi-orderable [RZ], [KR], and a recent paper of Gonzales-Meneses [G-M] constructs a bi-ordering on the pure braid groups of orientable surfaces $PB_n(M^2)$.

The goal of the present paper is to investigate the orderability of the fundamental groups of compact, connected 3-manifolds, a class we refer to as 3-*manifold groups*. We include nonorientable manifolds, and manifolds with boundary in the analysis. It will

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be seen that left-orderability is a rather common property in this class, but is by no means universal. After reviewing some general properties of orderable groups in §2, we begin our investigation of 3-manifold groups in §3, asking not only if such a group is left- or bi-orderable, but also if these properties hold virtually. In other words, we examine whether or not there is a finite cover of the manifold whose group is left- or bi-orderable. The following is one of our general results.

Theorem 1.1 *Suppose that M is a compact, connected and $\mathbb{P}^2$-irreducible 3-manifold.*
(1) *A necessary and sufficient condition that $\pi_1(M)$ be left-orderable is that either $\pi_1(M)$ is trivial or there exists a non-trivial homomorphism from $\pi_1(M)$ to a left-orderable group.*
(2) *If $\pi_1(M)$ is not virtually left-orderable, then M is closed, orientable and geometrically atoroidal, that is, there is no π_1-injective torus in M.*

Part (1) of this theorem follows from work of Howie and Short and an observation of Boileau. See theorem 3.2. Part (2) is a consequence of part (1) and a theorem of Luecke. See the discussion following conjecture 3.10.

Theorem 1.1 implies that a compact, connected, $\mathbb{P}^2$-irreducible 3-manifold M whose first Betti number $b_1(M)$ is larger than zero has a left-orderable fundamental group. This is, in fact, the generic case, as it is well-known that $b_1(M) > 0$ when M is neither a 3-ball nor a $\mathbb{Q}$-homology 3-sphere (cf. lemma 3.3). On the other hand, we will see below that not every such $\mathbb{Q}$-homology 3-sphere M has a left-orderable fundamental group. Nevertheless, it frequently does up to taking a finite index subgroup. Danny Calegari pointed out to us that this is the case when M is atoroidal and admits a transversely orientable taut foliation, owing to the existence of a faithful representation $\pi_1(M) \to Homeo_+(S^1)$ arising from Thurston's universal circle construction. More generally the following result holds (see §3).

Proposition 1.2 *Let M be an irreducible $\mathbb{Q}$-homology 3-sphere and $\hat{M} \to M$ the finite-sheeted cover corresponding to the commutator subgroup of $\pi_1(M)$. If there is a homomorphism $\pi_1(M) \to Homeo_+(S^1)$ with non-abelian image, then $\pi_1(\hat{M})$ is left-orderable. In particular if M is a $\mathbb{Z}$-homology 3-sphere, $\pi_1(M)$ is left-orderable.*

Corollary 1.3 (Calegari-Dunfield [CD]) *Let M be an irreducible, atoroidal $\mathbb{Q}$-homology 3-sphere which admits a transversely orientable taut foliation. If $\hat{M}$ is the covering described above, then $\pi_1(\hat{M})$ is left-orderable.* □

Corollary 1.4 *Let M be a Seifert fibred manifold which is also a $\mathbb{Z}$-homology 3-sphere. Then M is either homeomorphic to the Poincaré homology sphere (with $\pi_1(M)$ finite and non-trivial), or else $\pi_1(M)$ is left-orderable.*

Proof If M is a Seifert fibred manifold and a $\mathbb{Z}$-homology 3-sphere other than the Poincaré homology-sphere, it is $\mathbb{P}^2$-irreducible and either it is homeomorphic to S^3 or its base orbifold is hyperbolic (cf. §4). The bi-orderability of $\pi_1(M)$ is obvious in the former case, while in the latter we observe that the quotient of $\pi_1(M)$ by its centre is a non-trivial Fuchsian subgroup of $PSL_2(\mathbb{R}) \subset Homeo_+(S^1)$. In this case apply the previous proposition. $\square$

Background material on Seifert fibred spaces is presented in §4 while in §5 we examine the connection between orderability and codimension 1 objects such as laminations and foliations. After these general results, we focus attention on the special class of Seifert fibred 3-manifolds, possibly non-orientable, a convenient class which is well-understood, yet rich in structure. For this case we are able to supply complete answers.

Theorem 1.5 *The fundamental group of a compact, connected, Seifert fibred space M is left-orderable if and only if $M \cong S^3$ or one of the following two sets of conditions holds:*
(1) *$rank_{\mathbb{Z}} H_1(M) > 0$ and $M \not\cong \mathbb{P}^2 \times S^1$;*
(2) *M is orientable, the base orbifold of M is of the form $S^2(\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\pi_1(M)$ is infinite, and M admits a horizontal foliation.*

The definition of horizontal foliation is given in section 4. When applying this theorem, it is worth keeping in mind that Seifert manifolds whose first Betti number is zero and which have infinite fundamental group admit unique Seifert structures (see [Jc], theorem VI.17). We also remark that owing to the combined work of Eisenbud-Hirsch-Neumann [EHN], Jankins-Neumann [JN2] and Naimi [Na], it is known exactly which Seifert manifolds admit horizontal foliations (see theorem 4.2). This work and theorem 1.1 show that left-orderability is a much weaker condition than the existence of a horizontal foliation for Seifert manifolds of positive first Betti number.

Roberts and Stein have shown [RS] that a necessary and sufficient condition for an irreducible, non-Haken Seifert fibred manifold to admit a horizontal foliation is that its fundamental group act non-trivially (i.e. without a global fixed point) on $\mathbb{R}$, a condition which is (in this setting) equivalent to the left-orderability of the group (theorem 2.5). Since these Seifert manifolds have base orbifolds of the form $S^2(\alpha_1, \alpha_2, \alpha_3)$, theorem 1.5 can be seen as a generalization of their result.

Theorem 1.5 characterizes the Seifert manifold groups which are left-orderable. In order to characterize those which are bi-orderable, we must first deal with the same question for surface groups. It is well-known that free groups are bi-orderable. Moreover, it was observed by Baumslag that the fundamental group of an orientable surface is residually free, and therefore bi-orderable (see [Sm] and [Ba]). In §7 we give a new proof of the bi-orderability of closed orientable surface groups, and settle the orderability question for closed, nonorientable surface groups. This result also appears in [RoWi].

Theorem 1.6 *If M is any connected surface other than the projective plane or Klein bottle, then $\pi_1(M)$ is bi-orderable.*

In §8 we will use this result to prove

Theorem 1.7 *For the fundamental group of a compact, connected, Seifert fibred space M to be bi-orderable, it is necessary and sufficient that either*
(1) *M be homeomorphic to one of $S^3, S^1 \times S^2, S^1 \tilde{\times} S^2$ (the non-trivial 2-sphere bundle*

over the circle), a solid Klein bottle, or
(2) *M be the total space of a locally trivial, orientable circle bundle over a surface other than $S^2, \mathbb{P}^2$, or the Klein bottle $2\mathbb{P}^2$.*

Corollary 1.8 *The fundamental group of any compact Seifert fibred manifold is virtually bi-orderable.*

Proof A Seifert manifold is always finitely covered by an orientable Seifert manifold which is a circle bundle over an orientable surface. If that surface happens to be a 2-sphere, there is a further finite cover whose total space is either S^3 or $S^1 \times S^2$. $\square$

Seifert manifolds account for six of the eight 3-dimensional geometries. Of the two remaining geometries, hyperbolic and *Sol*, the latter is fairly simple to understand in terms of orderability properties. In §9 we prove the following theorem.

Theorem 1.9 *Let M be a compact, connected Sol manifold. Then*
(1) *$\pi_1(M)$ is left-orderable if and only if $\partial M \neq \emptyset$, or M is non-orientable, or M is orientable and a torus bundle over the circle.*
(2) *$\pi_1(M)$ is bi-orderable if and only if $\partial M \neq \emptyset$ but M is not a twisted I-bundle over the Klein bottle, or M is a torus bundle over the circle whose monodromy in $GL_2(\mathbb{Z})$ has at least one positive eigenvalue.*
(3) *$\pi_1(M)$ is virtually bi-orderable.*

In a final section we consider hyperbolic 3-manifolds. This is the geometry in which the orderability question seems to us to be the most difficult, and we have only partial results. We discuss a very recent example [RSS] of a closed hyperbolic 3-manifold whose fundamental group is not left-orderable. On the other hand, there are many closed hyperbolic 3-manifolds whose groups *are* LO – for example those which have infinite first homology (by theorem 1.1). This enables us to prove the following result.

Theorem 1.10 *For each of the eight 3-dimensional geometries, there exist closed, connected, orientable 3-manifolds with the given geometric structure whose fundamental groups are left-orderable. There are also closed, connected, orientable 3-manifolds with the given geometric structure whose groups are not left-orderable.*

This result seems to imply that geometric structure and orderability are not closely related. Nevertheless compact, connected hyperbolic 3-manifolds are conjectured to have finite covers with positive first Betti numbers, and if this is true, their fundamental groups are virtually left-orderable (cf. corollary 3.4 (1)). One can also ask whether they have finite covers with bi-orderable groups, though to put the relative difficulty of this question in perspective, note that nontrivial, finitely generated, bi-orderable groups have positive first Betti numbers (cf. theorem 2.7).

We close the introduction with several questions and problems arising from this study.

Question 1.11 Is the fundamental group of a compact, connected, orientable 3-manifold virtually left-orderable? What if the manifold is hyperbolic?

It is straightforward to argue that 3-manifold groups are virtually torsion free.

We saw in theorems 1.7 and 1.9 that the bi-orderability of the fundamental groups of Seifert manifolds and Sol manifolds can be detected in a straightforward manner. The same problem for hyperbolic manifolds appears to be much more subtle.

Question 1.12 Is there a compact, connected, orientable irreducible 3-manifold whose fundamental group is not virtually bi-orderable? What if the manifold is hyperbolic?

Problem 1.13 Find necessary and sufficient conditions for the fundamental group of a compact, connected 3-manifold which fibres over the circle to be bi-orderable. Equivalently, can one find bi-orderings of free groups or surface groups which are invariant under the automorphism corresponding to the monodromy of the fibration?

This problem is quickly dealt with in the case when the fibre is of non-negative Euler characteristic, so the interesting case involves fibres which are hyperbolic surfaces. When the boundary of the surface is non-empty, Perron and Rolfsen [PR] have found a sufficient condition for bi-orderability; for instance, the fundamental group of the figure eight knot exterior has a bi-orderable fundamental group.

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2 Ordered and bi-ordered groups.

We summarize a few facts about left-orderable (LO) groups, bi-orderable (O) groups and other algebraic matters. Good references are [MR], [Pa].

Proposition 2.1 *If G is left-orderable, then G is torsion-free.*

Proof If $g \neq 1$, we wish to show $g^p \neq 1$. Without loss of generality, $1 < g$. Then $g < g^2$, $g^2 < g^3$, etc. so an easy induction shows $1 < g^p$ for all positive p. □

A group G is *LO* if and only if there exists a subset $P \subset G$ (the positive cone) such that (1) $P \cdot P = P$ and (2) for every $g \neq 1$ in P, exactly one of g or g^{-1} belongs to P. Given such a P, the recipe $g < h$ if and only $g^{-1}h \in P$ is easily seen to define a left-invariant strict total order, and conversely such an ordering defines the set P as the set of elements greater than the identity. The group G is bi-orderable if and only if it admits a subset P satisfying (1), (2), and in addition (3) $gPg^{-1} \subset P$ for all $g \in G$.

The class of *LO* groups is closed under taking subgroups, extensions, directed unions, free products. The class of O groups is also invariant under taking subgroups, directed unions and free products ([V]), but not necessarily under extensions. An instructive example is the fundamental group of the Klein bottle:

$$G = \langle m, l \ ; \ lml^{-1} = m^{-1} \rangle.$$

This contains a normal subgroup $\mathbb{Z}$ generated by m, and the quotient $G/\mathbb{Z}$ is also an infinite cyclic group. Of course $\mathbb{Z}$ is bi-orderable, so the extension G of $\mathbb{Z}$ by $\mathbb{Z}$ is certainly left-orderable, by lemma 2.2 below. However, G is not biorderable, for if we had a biorder with $m > 1$ then it would follow that $1 < lml^{-1} = m^{-1} < 1$; if $m < 1$ a similar contradiction arises.

Lemma 2.2 *Let $f\colon G \to H$ be a surjective homomorphism of groups, with kernel K, and assume that both H and K are left-ordered, with positive cones P_H and P_K, respectively. Then the subset $P = P_K \cup f^{-1}(P_H)$ defines a left-invariant ordering on G by the rule $g < g' \Leftrightarrow g^{-1}g' \in P$. If H and K are bi-ordered, and if in addition P_K is normal in G, then this rule defines a bi-ordering of G.*

Proof Routine, and left to the reader. □

A *left action* of a group G on a set X is a homomorphism ϕ from G to the permutation group of X. For $g \in G$ and $x \in X$ we denote $\phi(g)(x)$ by $g(x)$. If $1 \in G$ is the only group element that acts as the identity on X, the action is said to be *effective.*

Theorem 2.3 (Conrad, 1959) *A group G is left-orderable if and only if it acts effectively on a linearly ordered set X by order-preserving bijections.*

Proof One direction is obvious, as a left-ordered group acts upon itself via multiplication on the left. On the other hand, assume G acts on X in such a way that for every $g \in G$, $x < y \Leftrightarrow g(x) < g(y)$. Let $\prec$ be some *well*-ordering of the elements of X, completely independent of the given ordering $<$ and of the G-action (such an order exists, by the axiom of choice). Compare $g \neq h \in G$ by letting $x_0 \in X$ be the smallest x, in the well-ordering $\prec$, such that $g(x) \neq h(x)$. Then say that $g < h$ or $h < g$ according as $g(x_0) < h(x_0)$ or $h(x_0) < g(x_0)$. This can easily be seen to be a left-invariant ordering of G. □

Thus, the group $Homeo_+(\mathbb{R})$ of order-preserving bijections is left-orderable; it acts effectively on $\mathbb{R}$ by definition. It follows that the universal covering group $\widetilde{SL_2}(\mathbb{R})$ of $PSL_2(\mathbb{R})$ is left-orderable, a fact first noted by Bergman [Be1], as it acts effectively and order-preservingly on the real line $\mathbb{R}$.

Next we state a classical result. A left-ordering of the group G is said to be *Archimedian* if for each $a, b \in G$ with $1 < a < b$, there is a positive integer n such that $b < a^n$.

Theorem 2.4 (Conrad 1959, Hölder 1902) *If the left-ordered group G is Archimedian, then the ordering is necessarily bi-invariant. Moreover G is isomorphic (in both the algebraic and order sense) with a subset of the additive real numbers, with the usual ordering. In particular, G is abelian.* □

This result simply implies that most interesting left-ordered groups are non-archimedian. The following offer alternative criteria for left-orderability; this is well-known to experts – see [Li] for a proof.

Theorem 2.5 *If G is a countable group, then the following are equivalent:*
(1) *G is left-orderable.*
(2) *G is isomorphic with a subgroup of $Homeo_+(\mathbb{R})$.*
(3) *G is isomorphic with a subgroup of $Homeo_+(\mathbb{Q})$.* $\square$

Theorem 2.6 (Burns-Hale [BH]) *A group is left-orderable if and only if every finitely generated subgroup has a non-trivial quotient which is left-orderable.* $\square$

We recall the definition due to Higman: a group is *locally-indicable* (LI) if every nontrivial finitely-generated subgroup has $\mathbb{Z}$ as a quotient. The following is also well-known to experts [Co], [BH], [MR].

Theorem 2.7 *If G is a bi-orderable group, then G is locally indicable. If G is locally indicable, then G is left-orderable. Neither of these implications is reversible.*

Proof If G is bi-ordered, consider a finitely generated subgroup $H = \langle h_1, \ldots, h_k \rangle$, with notation chosen so that $1 < h_1 < \ldots < h_k$. We recall that a subgroup C is called *convex* if $f < g < h, f \in C, h \in C \Rightarrow g \in C$. The convex subgroups of a left-ordered group are ordered by inclusion and closed under intersections and unions. Now, considering H itself as a finitely generated left-ordered group, we let K be the union of all convex subgroups of H which do not contain h_k. Then one can use bi-orderability and a generalization of the Conrad-Hölder theorem (or see [Co] for a more general argument) to show that K is normal in H, and the quotient H/K is isomorphic with a subgroup of $(\mathbb{R}, +)$. Being finitely generated, H/K is therefore isomorphic with a sum of infinite cyclic groups, and so there is a nontrivial homomorphism $H \to H/K \to \mathbb{Z}$, completing the first half of the theorem.

The second half follows directly from theorem 2.6, and the observation that $\mathbb{Z}$ is left-orderable. Finally, the fact that neither implication is reversible is discussed in the paragraph which follows. $\square$

Bergman [Be1] observed that even though $\widetilde{SL}_2(\mathbb{R})$ is left-orderable, it is not locally-indicable: for example, it contains the perfect infinite group $\langle x, y, z : x^2 = y^3 = z^7 = xyz \rangle$, which happens to be the fundamental group of a well-known homology sphere. The braid groups B_n, for $n > 4$ are further examples of LO groups which are not locally indicable, as their commutator subgroups $[B_n, B_n]$ are finitely generated and perfect. The braid groups B_3 and B_4, and the Klein bottle group provide examples of locally-indicable groups which are not bi-orderable. There is a characterization of those left-orderable groups which are locally indicable in [RR]. For instance for solvable groups [CK], and more generally, elementary amenable groups [Li], the concepts of left-orderability and local indicability coincide.

The following theorem of Farrell relates orderability with covering space theory.

Theorem 2.8 (Farrell [Fa]) *Suppose X is a locally-compact, paracompact topological space, and let p: $\widetilde{X} \to X$ the universal covering. Then $\pi_1(X)$ is left-orderable if and only if there is a topological embedding h: $\tilde{X} \to X \times \mathbb{R}$ such that $pr_1 h = p$.* $\square$

We conclude this section with certain facts about orderable groups, which makes orderability properties worthwhile knowing. Of particular interest are the deep properties of the group ring $\mathbb{Z}G$.

Theorem 2.9 (see eg. [Pa]) *If G is left-orderable, then $\mathbb{Z}G$ has no zero divisors, and only the units ng where n is a unit of $\mathbb{Z}$ and $g \in G$. The same is true for any integral domain R replacing $\mathbb{Z}$.* □

A proof is not difficult, the idea being to show that in a formal product, the largest (and smallest) terms in the product cannot be cancelled by any other term. The conclusions of this theorem are conjectured to be true for arbitrary torsion-free groups. For bi-orderable groups we know even more.

Theorem 2.10 (Mal'cev [Ma], B. Neumann [Ne]) *If G is bi-orderable then $\mathbb{Z}G$ embeds in a division algebra.* □

Theorem 2.11 (LaGrange, Rhemtulla [LR]) *Suppose G and H are groups with G left-orderable. Then G and H are isomorphic groups if and only if their group rings $\mathbb{Z}G$ and $\mathbb{Z}H$ are isomorphic as rings.* □

3 General remarks on ordering 3-manifold groups

3.1 Orderability

Lemma 3.1 (Vinogradov [V]) *A neccesary and sufficient condition for a free product $G = G_1 * G_2 * \ldots * G_n$ of groups to be left-orderable, respectively bi-orderable, is that each G_j has this property.* □

Recall that a compact, connected 3-manifold $M \neq S^3$ splits into a product of prime 3-manifolds under connected sum

$$M \cong M_1 \# M_2 \# \ldots \# M_n.$$

Clearly then $\pi_1(M) \cong \pi_1(M_1) * \pi_1(M_2) * \ldots * \pi_1(M_n)$ is LO, respectively O, if and only if each $\pi_1(M_j)$ has this property. Since the fundamental group of a prime, reducible 3-manifold is $\mathbb{Z}$, it suffices to investigate the orderability of the groups of irreducible 3-manifolds. The following is an extremely useful result special to 3-manifold groups (cf. theorem 1.1(1)).

Theorem 3.2 *Suppose that M is a compact, connected, $\mathbb{P}^2$-irreducible 3-manifold and that $\pi_1(M)$ is nontrivial. A necessary and sufficient condition that $\pi_1(M)$ be left-orderable is that there exists a homomorphism h: $\pi_1(M) \to L$ onto a nontrivial left-orderable group L.*

Recall that an irreducible manifold is $\mathbb{P}^2$-reducible if and only if it contains a 2-sided $\mathbb{P}^2$. This is equivalent to the manifold being non-orientable and there being a $\mathbb{Z}/2$ subgroup in its fundamental group ([Ep], Theorem 8.2).

The proof of theorem 3.2 is based on an argument of Howie and Short [HoSh], and an observation of Boileau. First we need a lemma.

Lemma 3.3 *If M is a compact 3-manifold and either M is closed and non-orientable or ∂M is nonempty but contains no S^2 or $\mathbb{P}^2$ components, then $H_1(M)$ is infinite.*

Proof We wish to show that the first Betti number $b_1(M)$ of M is at least one. Noting that closed 3-manifolds (even if nonorientable) have zero Euler characteristic, if W is the double of M, then $0 = \chi(W) = 2\chi(M) - \chi(\partial M)$, and so $2\chi(M) = \chi(\partial M)$. Our hypotheses imply that $H_3(M) = 0$ while each component of ∂M has a non-positive Euler charactersitic, thus

$$0 \geqslant \frac{1}{2}\chi(\partial M) = \chi(M) = \sum(-1)^j b_j(M) = 1 - b_1(M) + b_2(M)$$

and we conclude that $b_1(M) \geqslant 1 + b_2(M) \geqslant 1$. $\square$

Remark: A similar argument shows that if M is a nonorientable $\mathbb{P}^2$-irreducible 3-manifold (with or without boundary), then $b_1(M) > 0$. This will be used in section 6.

Proof of theorem 3.2 Necessity is obvious. For sufficiency, assume there is a surjection $h\colon \pi_1(M) \to L$, with L nontrivial left-orderable. We wish to show that $\pi_1(M)$ is left-orderable. Using the Burns-Hale characterization (theorem 2.6), it suffices to show that every nontrivial finitely generated subgroup H of $\pi_1(M)$ has a homomorphism onto a nontrivial left-ordered group. Consider such a group H and distinguish two cases. If H has finite index in $\pi_1(M)$, then $h(H)$ is a finite index subgroup of L and therefore nontrivial. So in this case we can just take the restriction of h to H.

Now suppose H has infinite index and let $p\colon \tilde{M} \to M$ be the corresponding covering space, i.e. $p_\#(\pi_1(\tilde{M}, \tilde{*})) = H$. Although $\tilde{M}$ is necessarily noncompact, by a theorem of Scott [Sc1], there is a compact submanifold $C \subset \tilde{M}$ whose fundamental group is isomorphic, via inclusion, with $\pi_1(\tilde{M})$. The manifold C must have nonempty boundary, otherwise it would be all of $\tilde{M}$. Suppose that $S \subset \partial C$ is a 2-sphere. Since M is irreducible, so is $\tilde{M}$ [MSY] (see [Du], [Ha]), and therefore S bounds a 3-ball B in $\tilde{M}$. We claim that $B \cap C = S$, for otherwise we would have $C \subset B \subset \tilde{M}$, contradicting that the inclusion of C in $\tilde{M}$ induces an isomorphism of nontrivial groups. Thus we may attach B to C without changing the property that $i_*\colon \pi_1(C) \to \pi_1(\tilde{M})$ is an isomorphism. Therefore we may assume ∂C contains no 2-sphere components. Next we wish to show that no component of ∂C is a projective plane. If there were such a component, it would contain a loop α which reverses orientation of $\tilde{M}$, and hence is nontrivial in $\pi_1(\tilde{M})$. On the other hand, since it lies in the projective plane, $\alpha^2 = 1$; which would imply that $\pi_1(\tilde{M})$, and therefore $\pi_1(M)$ has an element of order 2, which is not allowed. We now have that ∂C is nonempty, but contains no spheres or projective planes. By lemma 3.3, $H_1(C)$ is infinite, and therefore maps onto $\mathbb{Z}$. Preceding this homomorphism by the Hurewicz map $\pi_1(C) \to H_1(C)$ gives the required homomorphism of H onto $\mathbb{Z}$. $\square$

Corollary 3.4 *Let M be a compact, connected, prime 3-manifold, possibly with boundary.*
(1) *If M is orientable with $H_1(M)$ infinite, then $\pi_1(M)$ is left-orderable.*
(2) *If M is non-orientable then $\pi_1(M)$ is left-orderable if and only if it contains no element of order 2.*

Proof If M is reducible, its group is $\mathbb{Z}$, so the corollary holds. On the other hand if we assume that M is irreducible, then part (1) follows directly from theorem 3.2. Part (2) does as well once we consider the remark following the statement of theorem 3.2. □

Corollary 3.5 *Let $G = \pi_1(S^3 \setminus K)$ be a knot or link group. Then G is left-orderable.*

Proof The only point to observe is that the group of a split link is a free product of the groups of non-split links (cf. lemma 3.1), whose complements are irreducible. □

Remark A similar argument to that in the proof of theorem 3.2 shows that in corollaries 3.4 and 3.5, we may replace "left-orderable" by the stronger condition "locally indicable". Therefore the only compact prime 3-manifolds which can have LO but not LI fundamental groups are those with finite first homology. Bergman's example is just such a manifold.

We saw above that compact, connected, orientable, irreducible 3-manifolds with positive first Betti numbers have left-orderable groups. Such manifolds are Haken. On the other hand, not all Haken 3-manifolds have left-orderable groups (see eg. theorem 1.5). The simplest examples were constructed by Boileau, Short and Wiest.

Example 3.6 (Boileau, Short and Wiest) Let X be the exterior of a trefoil knot $K \subset S^3$ and let μ, ϕ denote, respectively, the meridional slope on ∂X and the slope corresponding to a fibre of the Seifert structure on X. Fix a base point $* \in \partial X$ and oriented representative curves C_1, C_2 for μ and ϕ based at $*$. The group $\pi_1(X; *)$ has a presentation $\langle x, y \mid x^2 = y^3 \rangle$ where xy^{-1} represents the class of C_1 while x^2 represents that of C_2. Since C_1 and C_2 intersect once algebraically, there is a homeomorphism $f\colon \partial X \to \partial X$ which switches them. The manifold $M = X \cup_f X$ is Haken, because the separating torus is incompressible. We claim that its fundamental group is not left-orderable.

Assume to the contrary that $<$ is a left-order on

$$\pi_1(M; *) = \langle x_1, y_1, x_2, y_2 \mid x_1^2 = y_1^3, x_2^2 = y_2^3, x_1^2 = x_2 y_2^{-1}, x_2^2 = x_1 y_1^{-1} \rangle.$$

Without loss of generality, $x_1 > 1$. The relation $y_1^3 = x_1^2$ implies that $y_1 > 1$ as well. Hence $x_1 > y_1^{-1}$, or equivalently, $x_1^2 > x_1 y_1^{-1}$. If $x_2 > 1$, a similar argument shows $x_2^2 > x_2 y_2^{-1}$. But then $x_1^2 > x_1 y_1^{-1} = x_2^2 > x_2 y_2^{-1} = x_1^2$, a contradiction. Hence $x_2 < 1$. Now $y_1^2 > x_1^{-1}$ implies $x_1^2 = y_1^3 > y_1 x_1^{-1}$, while $x_2 < 1$ implies $x_2^2 = y_2^3 < y_2 x_2^{-1}$. Thus $x_1^2 > y_1 x_1^{-1} = x_2^{-2} > x_2 y_2^{-1} = x_1^2$, another contradiction. It follows that there is no left-order on $\pi_1(M)$.

3.2 An application to mappings between 3-manifolds

Now that we have an example of a 3-manifold whose group is infinite and torsion-free, yet not left-orderable (there are many others), it is appropriate to point out an application of theorem 1.1. An important question in 3-manifold theory is whether, given two closed oriented 3-manifolds M and N, there exists a degree one map $M \to N$. Or, more generally, a map of nonzero degree. The following can be viewed as providing a new "obstruction" to the existence of such a map.

Theorem 3.7 *Let M and N be closed, oriented 3-manifolds, with M prime. Suppose $\pi_1(N)$ is nontrivial and left orderable, but $\pi_1(M)$ is NOT left orderable. Then any mapping $M \to N$ has degree zero.*

Proof Assume the hypotheses. Being prime and orientable, M is either irreducible or $S^2 \times S^1$, but the latter possibility is excluded by hypothesis. Suppose there were a mapping $M \to N$ of nonzero degree. According to the lemma below, the induced map $\pi_1(M) \to \pi_1(N)$ would be nontrivial. But then, by theorem 1.1, $\pi_1(M)$ would be left-orderable, a contradiction. $\square$

Lemma 3.8 *If $f : M \to N$ is a mapping of nonzero degree, then $f_*(\pi_1(M))$ has finite index in $\pi_1(N)$.*

Proof Let $p : \tilde{N} \to N$ denote the cover corresponding to $f_*(\pi_1(M))$, so there is a lift $\tilde{f} : M \to \tilde{N}$. Now $\tilde{N}$ must be compact, otherwise $H_3(\tilde{N}) = 0$, and since f factors through $\tilde{N}$ its degree would be zero. Thus the covering is finite-sheeted, and the index is finite. $\square$

3.3 Virtual orderability

Though a 3-manifold group may not be left-orderable, it seems likely that it contains a finite index subgroup which is.

Lemma 3.9 *A neccesary and sufficient condition for a free product $G = G_1 * G_2 * \ldots * G_n$ of groups to be virtually left-orderable (resp. virtually bi-orderable) is that each G_j have this property.*

Proof Let $j \in \{1, 2, \ldots, n\}$, and let $H \subseteq G$ be a LO subgroup of finite index. We may assume that H is normal (after replacing H by the intersections of all its conjugates in G). Thus we have a homomorphism ϕ: $G \to F = G/H$ where F is a finite group and $\ker(\phi) = H$ is left-orderable. Clearly the kernel of the composition $G_j \hookrightarrow G \to F$ is left-orderable (being contained in H), and is of finite index in G_j. Thus G_j is virtually left-orderable.

On the other hand if each G_j is virtually left-orderable, there are surjective homomorphisms ϕ_j: $G_j \to F_j$ where $F_1, F_2, \ldots, F_n$ are finite groups and $\ker(\phi_j) < G_j$ is left-orderable. By the Kurosh subgroup theorem [ScWa], the kernel of the obvious homomorphism $G_1 * G_2 * \ldots * G_n \to F_1 \times F_2 \times \ldots \times F_n$ is a free product of a free group and groups isomorphic to $\ker(\phi_1), \ldots, \ker(\phi_n)$. This finite-index subgroup is left-orderable by lemma 3.1.

A similar arguments shows the analoguous statement for *bi*-orderable groups. $\square$

We are therefore reduced to investigating the virtual orderability properties of the group of a prime 3-manifold M. It is clear that we may restrict our attention to prime 3-manifolds which are irreducible. We recall the following variant of a conjecture of Waldhausen.

Conjecture 3.10 If M is a compact, connected, $\mathbb{P}^2$-irreducible 3-manifold with infinite fundamental group, then there is a finite cover $\tilde{M} \to M$ with $b_1(\tilde{M}) > 0$.

If this conjecture turns out to be true, then theorem 3.2 implies that that any prime $\mathbb{P}^2$-irreducible 3-manifold M has a virtually left-orderable group. While examining this possibility, we may as well assume that the manifold is closed and orientable (corollary 3.4). John Luecke has shown [Lu] that Conjecture 3.10 holds for any such M which contains a π_1-injective torus, and therefore its fundamental group is virtually left-orderable. We may therefore assume that M is geometrically atoroidal as well as being irreducible and closed. Such manifolds are known to be either Seifert fibred or simple, and conjecturally Seifert or hyperbolic. We will see in §8 that the groups of Seifert manifolds are virtually bi-orderable, but we do not know if this holds for hyperbolic manifolds.

We remark that by theorem 2.7, if M is a compact, connected, $\mathbb{P}^2$-irreducible 3-manifold which has a virtually bi-orderable fundamental group, then this group is virtually locally indicable. Hence it has a virtually positive first Betti number. This puts the relative difficulty of the virtual bi-orderability of 3-manifold groups in perspective.

Next we apply theorem 3.2 to prove proposition 1.2. We begin with a simple lemma pointed out to us by Danny Calegari.

Lemma 3.11 *Let Γ be a group such that $H_2(\Gamma) = 0$. Suppose that $1 \to A \to \tilde{G} \to G \to 1$ is a central extension of a group G by a group A. If $\rho : \Gamma \to G$ is a homomorphism, then the restriction $\rho|[\Gamma, \Gamma]$ lifts to a homomorphism $[\Gamma, \Gamma] \to \tilde{G}$.*

Proof Fix $\gamma = \Pi_{i=1}^n [\alpha_i, \beta_i] \in [\Gamma, \Gamma]$ and let $\tilde{a}_i, \tilde{b}_i$ be arbitrary lifts of $\rho(\alpha_i), \rho(\beta_i)$ to $\tilde{G}$. The centrality of the extension shows that $\tilde{\gamma} := \Pi_{i=1}^n [\tilde{a}_i, \tilde{b}_i]$ is independent of our choice of lifts. In fact $\tilde{\gamma}$ is independent of the way we expressed γ as a product of commutators. Equivalently, we claim that if $\Pi_{i=1}^n [\alpha_i, \beta_i] = 1$, then $\Pi_{i=1}^n [\tilde{a}_i, \tilde{b}_i] = 1 \in \tilde{G}$. Once we show this, the correspondence $\gamma \mapsto \tilde{\gamma}$ provides the desired lift of $\rho|[\Gamma, \Gamma]$.

Let $1 \to R \to F \xrightarrow{\phi} \Gamma \to 1$ be a free presentation of Γ and fix a lift $\tilde{\phi} : F \to \tilde{G}$ of $\rho \circ \phi$. Then $\tilde{\phi}(R) \subset A$ lies in the centre of $\tilde{G}$. Choose $x_i, y_i \in F$ which are sent to α_i, β_i by ϕ. Then by construction, $\Pi_{i=1}^n [x_i, y_i] \in R \cap [F, F]$. On the other hand Hopf's formula [HiSt] shows that $0 = H_2(\Gamma) = (R \cap [F, F])/[F, R]$, and so $\Pi_{i=1}^n [x_i, y_i] = \Pi_{j=1}^m [f_j, r_j]$ for some $f_j \in F$ and $r_j \in R$. Then

$$\Pi_{i=1}^n [\tilde{a}_i, \tilde{b}_i] = \Pi_{i=1}^n [\tilde{\phi}(x_i), \tilde{\phi}(y_i)] = \Pi_{j=1}^m [\tilde{\phi}(f_j), \tilde{\phi}(r_j)] = 1$$

since $\tilde{\phi}(r_j)$ is contained in the centre of $\tilde{G}$ for each j. This completes the proof. $\square$

Proof of proposition 1.2 Recall that M is an irreducible $\mathbb{Q}$-homology 3-sphere and that $\hat{M} \to M$ is the cover corresponding to the commutator subgroup of $\pi_1(M)$. We are given a homomorphism $\rho\colon \pi_1(M) \to Homeo_+(S^1)$ whose image is not abelian and we want to deduce that $\pi_1(\hat{M})$ is left-orderable.

Consider the central $\mathbb{Z}$ extension

$$1 \to \mathbb{Z} \to \widetilde{Homeo}_+(S^1) \to Homeo_+(S^1) \to 1$$

where $\widetilde{Homeo}_+(S^1)$ is the universal covering group of $Homeo_+(S^1)$. This covering group can be identified with the subgroup of $Homeo_+(\mathbb{R})$ consisting of period 1 homeomorphisms in such a way that its central $\mathbb{Z}$ subgroup corresponds to translations $T_n : x \mapsto x + n, n \in \mathbb{Z}$. Since M is irreducible, $H_2(\pi_1(M)) \cong H_2(M)$, while $H_2(M) \cong 0$ since M is a $\mathbb{Q}$-homology 3-sphere. Hence the previous lemma implies that the restriction of ρ to $\pi_1(\hat{M})$ lifts to a homomorphism $\pi_1(\hat{M}) \to \widetilde{Homeo}_+(S^1) \subset Homeo_+(\mathbb{R})$. Since ρ has non-abelian image, the image of the lifted homomorphism will not be the trivial group. Theorem 3.2 now implies the desired conclusion. $\square$

Example 3.12 Let M_K denote the exterior of the figure 8 knot K. For each extended rational number $\frac{p}{q} \in \mathbb{Q} \cup \{\frac{1}{0}\}$ let $M_K(\frac{p}{q})$ be the $\frac{p}{q}$-Dehn filling of M_K, that is $M_K(\frac{p}{q})$ is the manifold obtained by attaching a solid torus V to M_K in such a way that the meridian of V wraps p times meridionally around K and q times longitudinally. Each of these manifolds is irreducible and is a $\mathbb{Q}$-homology 3-sphere if and only if $\frac{p}{q} \neq 0$. We will show that for $-4 < \frac{p}{q} < 4$, $\pi_1(M_K(\frac{p}{q}))$ admits a representation to $PSL_2(\mathbb{R})$ with non-abelian image and hence is virtually left-orderable (when $\frac{p}{q} = 0$ apply Corollary 3.4). We remark that Nathan Dunfield has announced that each $M_K(\frac{p}{q})$, $\frac{p}{q} \neq \infty$, has a finite cover with a positive first Betti number and so it follows that each Dehn filling of M_K has a virtually left-orderable fundamental group.

There is a presentation of the form

$$\pi_1(M_K) = \langle x, y \mid wx = yw \rangle$$

where x represents a meridian of K and $w = xy^{-1}x^{-1}y$. Given $s \geq \frac{1+\sqrt{5}}{2}$ set $t = \frac{1}{2(s-s^{-1})}(1 + \sqrt{(s-s^{-1})^4 + 2(s-s^{-1})^2 - 3}) \in \mathbb{R}$. The reader can verify that there is a representation $\phi_s : \pi_1(M_K) \to SL_2(\mathbb{R})$ such that

$$\phi_s(x) = \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}, \quad \phi_s(y) = \begin{pmatrix} (\frac{s+s^{-1}}{2}) + t & (\frac{s-s^{-1}}{2}) + t \\ (\frac{s-s^{-1}}{2}) - t & (\frac{s+s^{-1}}{2}) - t \end{pmatrix}.$$

It is simple to see that each ϕ_s has a non-abelian image in $PSL_2(\mathbb{C})$ and that ϕ_s is reducible if and only if $s = \frac{1+\sqrt{5}}{2}$.

The pair $\mu = x, \lambda = yx^{-1}y^{-1}x^2y^{-1}x^{-1}y$ represent meridian, longitude classes for K. Set $A_s = \phi_s(\mu), B_s = \phi_s(\lambda)$. As A_s is diagonal but not $\pm I$ and $[A_s, B_s] = I$, B_s is also diagonal. Let $\zeta(A_s), \zeta(B_s) \in \mathbb{R}$ be the $(1,1)$-entries of A_s, B_s. Then $\zeta(A_s) = s$ while direct calculation yields

$$\zeta(B_s) = \frac{1}{2s^4}((s^8 - s^6 - 2s^4 - s^2 + 1) + (s^4 - 1)\sqrt{s^8 - 2s^6 - s^4 - 2s^2 + 1})$$

Now ϕ_s induces a homomorphism $\pi_1(M_K(p/q)) \to PSL_2(\mathbb{R})$ if and only if $\zeta(A_s)^p\zeta(B_s)^q = \pm 1$, or equivalently

$$-\frac{ln|\zeta(B_s)|}{ln|\zeta(A_s)|} = \frac{p}{q}.$$

Thus we must examine the range of the function

$$g : [\frac{1+\sqrt{5}}{2}, \infty) \to \mathbb{R}, \quad s \mapsto -\frac{ln|\zeta(B_s)|}{ln|\zeta(A_s)|}.$$

Since $\phi_{\frac{1+\sqrt{5}}{2}}$ is reducible and λ lies in the commutator subgroup of $\pi_1(M_K)$, $\zeta(B_{\frac{1+\sqrt{5}}{2}}) = 1$ and therefore $ln|\zeta(B_{\frac{1+\sqrt{5}}{2}})| = 0$. On the other hand $\zeta(A_{\frac{1+\sqrt{5}}{2}}) = \frac{1+\sqrt{5}}{2} > 1$ so that $ln|\zeta(A_{\frac{1+\sqrt{5}}{2}})| > 0$. It follows that $g(\frac{1+\sqrt{5}}{2}) = 0$.

Next observe that $\lim_{s\to\infty} \zeta(A_s)^{-4}\zeta(B_s) = \lim_{s\to\infty} s^{-4}\zeta(B_s) = 1$. Therefore

$$\lim_{s\to\infty}(-4ln|\zeta(A_s)| + ln|\zeta(B_s)|) = 0,$$

which yields $\lim_{s\to\infty} g(s) = 4$. Hence the range of g contains $[0, 4)$ and so for each rational $\frac{p}{q}$ in this interval, there is at least one $s(\frac{p}{q}) \in (\frac{1+\sqrt{5}}{2}, \infty)$ such that $\phi_{s(\frac{p}{q})}$ factors through $\pi_1(M_K(\frac{p}{q}))$. Further the image of this representation is non-abelian. Our argument is completed by observing that the amphicheirality of M_K implies that if $\pi_1(M_K(\frac{p}{q}))$ admits a non-abelian representation to $PSL_2(\mathbb{R})$, then so does $\pi_1(M_K(-\frac{p}{q}))$.

4 Seifert fibre spaces

In this section we develop some background material on Seifert fibred spaces which will be used later in the paper. This important class of 3-manifolds was introduced by Seifert [Seif] in 1933, and later extended to include singular fibres which reverse orientation. We adopt the more general definition, as in Scott [Sc2]. A Seifert fibred space is a 3-manifold M which is foliated by circles. It is assumed that each leaf C, called a *fibre*, has a closed tubular neighbourhood $N(C)$ consisting of fibres. If C reverses orientation in M, then $N(C)$ is a *fibred solid Klein bottle.* A specific model is given by

$$(D^2 \times I)/\{(x,1) = (r(x), 0)\}$$

where $D^2 \subset \mathbb{C}$ is the unit disk, r: $D^2 \to D^2$ is a reflection (e.g. complex conjugation), and the foliation is induced from the I-factors in $D^2 \times I$. Note that most fibres wind twice around $N(C)$, but there is also an annulus consisting of *exceptional* fibres, each of which winds around $N(C)$ once.

If C preserves orientation, then $N(C) \cong S^1 \times D^2$ is a *fibred solid torus.* In this case the fibre preserving homeomorphism classes of such objects are parameterised by an integer $\alpha \geqslant 1$ and the $\pm$ class (mod α) of an integer q coprime with α. Specific models are represented by

$$(D^2 \times I)/\{(x,1) = (e^{\frac{2\pi i q}{\alpha}}x, 0)\}$$

endowed with the foliation by circles induced from the I-factors. The fibre C_0 corresponding to $\{0\} \times I$ winds once around $N(C)$, while the others wind α times. If $C = C_0$

we define the *index* of C to be α, otherwise 1. Note that the index of an orientation preserving fibre C is well-defined. Such a fibre is referred to as *exceptional* if its index is larger than 1.

The reader will verify that the space of leaves in $N(C)$ is always a 2-disk, and therefore the space of leaves in M, called the *base space*, is a surface B. There is more structure inherent in B, however. Indeed, it is the underlying space of a 2-dimensional orbifold $\mathcal{B}$, called the *base orbifold* of M, whose singular points correspond to the exceptional fibres C of the given Seifert structure. If C preserves orientation, then the associated point in B is a cone point, lying in $\mathrm{int}(B)$, whose order equals the index of C. If C reverses orientation, then it corresponds to a reflector point in ∂B, which in turn lies on a whole curve of reflector points in B. The base space B will also be written $|\mathcal{B}|$. There is a short exact sequence (see, for instance, lemma 3.2 of [Sc2])

$$1 \to K \to \pi_1(M) \to \pi_1^{orb}(\mathcal{B}) \to 1 \tag{4.1}$$

where K is the cyclic subgroup of $\pi_1(M)$ generated by a regular fibre and $\pi_1^{orb}(\mathcal{B})$ is the orbifold fundamental group of $\mathcal{B}$ ([Th1], Chapter 13).

In the case that M is orientable, the singularities of $\mathcal{B}$ are cone points lying in the interior of B. We shall say that $\mathcal{B}$ is of the form $B(\alpha_1, \alpha_2, \ldots, \alpha_n)$ where $\alpha_1, \alpha_2, \ldots, \alpha_n \geqslant 2$ are the indices of the exceptional fibres. Note that in this case ∂M is foliated by regular fibres and so consists of tori.

Following are some well-known facts about Seifert fibred spaces which will be useful.

Proposition 4.1 *Suppose M is a compact, connected Seifert fibred space and denote by $h \in \pi_1(M)$ a class corresponding to a regular (i.e. non-exceptional) fibre.*
(1) *If h has finite order, then M is orientable and finitely covered by S^3. In particular, $\pi_1(M)$ is finite.*
(2) *If $h = 1$, then $M \cong S^3$.*
(3) *If M is reducible, then $M = S^1 \times S^2$ or $S^1 \tilde{\times} S^2$ or $\mathbb{P}^3 \# \mathbb{P}^3$. The first two have (bi-orderable) group $\mathbb{Z}$ and Betti number $b_1(M) = 1$. However $\mathbb{P}^3 \# \mathbb{P}^3$ has $b_1 = 0$, group $\mathbb{Z}/2 * \mathbb{Z}/2$ (which is not left-orderable), and base orbifold $\mathbb{P}^2$.*
(4) *If M is nonorientable with 2-torsion in $\pi_1(M)$, then $M = \mathbb{P}^2 \times S^1$ with base orbifold $\mathbb{P}^2$. Its group $\mathbb{Z}/2 \times \mathbb{Z}$ is not left-orderable.*
(5) *If $\pi_1(M) \cong \mathbb{Z}$, then $M = S^1 \times S^2$ or $S^1 \tilde{\times} S^2$ or a solid torus or solid Klein bottle.*

Proof (1) Let $p : \tilde{M} \to M$ be the universal cover of M. If h has finite order in $\pi_1(M)$, then the inverse image of each fibre in M is a circle in $\tilde{M}$. In this way there is a Seifert fibring of $\tilde{M}$ with base orbifold $\tilde{\mathcal{B}}$ say and a commutative diagram

$$\begin{array}{ccc} \tilde{M} & \longrightarrow & \tilde{\mathcal{B}} \\ \downarrow & & \downarrow \\ M & \longrightarrow & \mathcal{B} \end{array}$$

where $\tilde{\mathcal{B}} \to \mathcal{B}$ is an orbifold covering. The simple-connectivity of $\tilde{M}$ implies that $\pi_1^{orb}(\tilde{\mathcal{B}}) = \{1\}$ (cf. exact sequence (4.1)) and therefore Riemann's uniformization theorem and theorem 2.3 of [Sc2] imply that $\tilde{\mathcal{B}}$ is either a contractible surface without cone points or one of $S^2, S^2(p)$ or $S^2(p,q)$ where $\gcd(p,q)=1$. The first case is ruled out as otherwise $\tilde{M} \cong |\tilde{\mathcal{B}}| \times S^1 \simeq S^1$ is not 1-connected. In the latter three cases, $\tilde{M}$ is a union of two solid tori and therefore must be the 3-sphere. Hence the fundamental group of M is finite. Since S^3 admits no fixed-point free orientation reversing homeomorphism, M is orientable.

(2) Next assume that $h=1$. By (1) $\tilde{M} \cong S^3$ and $\tilde{\mathcal{B}}$ is either $S^2, S^2(p)$ or $S^2(p,q)$ where $\gcd(p,q)=1$. We also know that $\pi_1(M)$ is finite and M is orientable. By hypothesis, the inclusion of each fibre of M lifts to an inclusion of the fibre in $\tilde{M}$. It follows that $\pi_1(M)$ acts freely on the components of the inverse image of any fibre of M. Thus the induced action of $\pi_1(M)$ on $|\tilde{\mathcal{B}}| \cong S^2$ is free and therefore $\pi_1(M)$ is a subgroup of $\mathbb{Z}/2$. We will show that $\pi_1(M) \not\cong \mathbb{Z}/2$.

Assume otherwise and observe that since $\pi_1(M)$ freely permutes the exceptional fibres of the Seifert structure on $\tilde{M}$, the only possibility is for $\tilde{\mathcal{B}} = S^2$. Exact sequence (4.1) yields $\pi_1^{orb}(\mathcal{B}) \cong \pi_1(M) \cong \mathbb{Z}/2$ and so M is a locally trivial S^1-bundle over $\mathbb{P}^2$. Splitting $\mathcal{B}$ into the union of a Möbius band and a 2-disk shows that M is a Dehn filling of the twisted I-bundle over the Klein bottle. A homological calculation then shows that the order of $H_1(M)$ is divisible by 4. But this contradicts our assumption that $\pi_1(M) \cong \mathbb{Z}/2$. Thus $\pi_1(M) = \{1\}$ and so $M = \tilde{M}/\pi_1(M) = \tilde{M} \cong S^3$.

(3) Suppose that M is reducible and let $S \subset M$ be an essential 2-sphere. The universal cover $\tilde{M}$ of M is also reducible as otherwise a 3-ball bounded by an innermost lift of S to $\tilde{M}$ projects to a ball bounded by S. Now the interior of the universal cover of a Seifert fibred space is either $S^3, \mathbb{R}^3$ or $S^2 \times \mathbb{R}$ (see eg. [Sc2, Lemma 3.1]) and therefore the interior of $\tilde{M}$ is homeomorphic to $S^2 \times \mathbb{R}$. By part (1) h has infinite order in $\pi_1(M)$ and it is not hard to see that the quotient of $S^2 \times \mathbb{R}$ by some power of h is $S^2 \times S^1$. Thus M itself is finitely covered by $S^2 \times S^1$ and so is one of $S^1 \times S^2, \mathbb{P}^3 \# \mathbb{P}^3, S^1 \tilde{\times} S^2$, or $S^1 \times \mathbb{P}^2$ [Tlf]. If $M \cong S^1 \times \mathbb{P}^2$, then the fact that $H_2(S^1 \times \mathbb{P}^2) = 0$ implies that S is separating and consideration of $\pi_1(S^1 \times \mathbb{P}^2) \cong \mathbb{Z} \oplus \mathbb{Z}/2$ implies that it bounds a simply-connected submanifold A of $S^1 \times \mathbb{P}^2$. Hence A lifts to $\tilde{M} \cong S^2 \times \mathbb{R} \subset \mathbb{S}^3$. It follows that A is a 3-ball and therefore $M \not\cong S^1 \times \mathbb{P}^2$.

(4) Suppose that M is nonorientable with 2-torsion in $\pi_1(M)$ and let $\tilde{M}$ be its universal cover. The group $\pi_1(M)$ is infinite by part (1) and so $\tilde{M}$ is non-compact. In particular $H_q(\tilde{M}) = 0$ if $q \geq 3$. If $\pi_2(M) = 0$, then $H_q(\tilde{M}) = 0$ for all q and so $\tilde{M}$, being simply-connected, is contractible. But then the quotient of $\tilde{M}$ by a cyclic group of order two $\mathbb{Z}/2 \subset \pi_1(M)$ would be a $K(\mathbb{Z}/2, 1)$, which is impossible as $\mathbb{Z}/2$ has infinite cohomological dimension. Hence $\pi_2(\tilde{M}) = \pi_2(M) \neq 0$, which implies that $\tilde{M} \cong S^2 \times \mathbb{R}$ and M is closed (cf. the proof of part (3)). Amongst the four closed manifolds covered by $S^2 \times \mathbb{R}$ only $\mathbb{P}^2 \times S^1$ satisfies the hypotheses of (4).

(5) Suppose that $\pi_1(M) \cong \mathbb{Z}$. If $\partial M \neq \emptyset$ it contains a compressible torus or Klein bottle. By parts (3) and (4) M is $\mathbb{P}^2$-irreducible and therefore M is either a solid torus or a solid Klein bottle. On the other hand if $\partial M = \emptyset$ and M is orientable, any non-separating closed, connected, orientable surface in M (which exists since $b_1(M) = 1$) may be compressed down to a non-separating 2-sphere. Thus by part (3) M is $S^1 \times S^2$. This implies that if $\partial M = \emptyset$ and M is non-orientable, then $M \cong S^1 \tilde{\times} S^2$ (cf. the

argument in part (3)). $\square$

Consider a closed, connected, oriented Seifert manifold M. A useful notation for such manifolds appears in [EHN] which we describe next. The base orbifold of M is of the form $B(\alpha_1, \alpha_2, \ldots, \alpha_n)$ where B is a closed surface and $\alpha_1, \alpha_2, \ldots, \alpha_n \geqslant 2$. As is well-known, B is determined by

$$g = \begin{cases} 1 - \frac{\chi(B)}{2} & \text{if } B \text{ is orientable} \\ \chi(B) - 2 & \text{if } B \text{ is non-orientable.} \end{cases}$$

When $n = 0$, $p\colon M \to B$ is an S^1-bundle whose total space is oriented, and so M is completely determined by g and an integer b measuring the obstruction to the existence of a cross-section. An explicit description of b is obtained as follows. Let $D \subset \mathrm{int}(B)$ be a 2-disk and set $B_0 = B \setminus \mathrm{int}(D), M_0 = p^{-1}(B_0)$, so that M is constructed from M_0 by attaching the solid torus $p^{-1}(D)$, i.e. M is obtained from M_0 by a Dehn filling. The bundle $p_0\colon M_0 = p^{-1}(B_0) \to B_0$ is uniquely determined by the fact that its total space is orientable, and it can be shown that p_0 admits a section s. The orientation on M determines orientations on $s(\partial B_0)$ and a circle fibre H on $p_0^{-1}(\partial B_0)$, and hence a homology basis $[s(\partial B_0)], [H]$ for $H_1(p_0^{-1}(\partial B_0))$, well-defined up to a simultaneous change of sign. Then b is the unique integer such that the meridional slope of $p^{-1}(D)$ corresponds to $\pm([s(\partial B_0)] + b[H]) \in H_1(p_0^{-1}(\partial B_0))$.

When $n > 0$ we proceed similarly. Let $C_1, C_2, \ldots, C_n$ be the exceptional fibres in M, C_0 a regular fibre, and $x_0, x_1, x_2, \ldots, x_n \in B$ the points to which they correspond. Choose disjoint 2-disks $D_0, D_1, D_2, \ldots, D_n \subset B$ where $x_j \in \mathrm{int}(D_j)$ and set $B_0 = B \setminus \bigcup_i \mathrm{int}(D_i), M_0 = p^{-1}(B_0)$. The S^1-bundle $M_0 \to B_0$ admits a section s and as in the last paragraph, the homology classes of the meridional slopes of the solid tori $p^{-1}(D_j)$ are of the form $\pm(\alpha_j[s(\partial D_j)] + \beta_j[H_j])$ where α_j is the index of C_j. In fact there is a unique choice of s, up to vertical homotopy, satisfying $0 < \beta_j < \alpha_j$ for $j = 1, 2, \ldots, n$. Make this choice and set $b = \beta_0$. Then M both determines and is determined by the integers g, b and the rational numbers $\frac{\beta_1}{\alpha_1}, \frac{\beta_2}{\alpha_2}, \ldots, \frac{\beta_n}{\alpha_n} \in (0, 1)$. Conversely given such a sequence of numbers we may construct a closed, connected, oriented Seifert manifold which realizes them. In the notation of [EHN],

$$M = M(g; b, \frac{\beta_1}{\alpha_1}, \frac{\beta_2}{\alpha_2}, \ldots, \frac{\beta_n}{\alpha_n}).$$

The fundamental group of this manifold is given by

$$\begin{aligned} \pi_1(M) = \langle a_1, b_1, ..., a_g, b_g, \gamma_1, \ldots, \gamma_n, h \mid h \ \textit{central}, \ \ \gamma_j^{\alpha_j} = h^{-\beta_j} \ \ (j = 1, ..., n) \\ [a_1, b_1]...[a_g, b_g]\gamma_1...\gamma_n = h^b \rangle \end{aligned}$$

when $g \geqslant 0$, and

$$\begin{aligned} \pi_1(M) = \langle a_1, ..., a_{|g|}, \gamma_1, \ldots, \gamma_n, h \mid a_j h a_j^{-1} = h^{-1} \ \ (j = 1, ..., |g|), \ \ \gamma_j^{\alpha_j} = h^{-\beta_j}, \\ \gamma_j h \gamma_j^{-1} = h \ \ (j = 1, ..., n), \ \ a_1^2 ... a_{|g|}^2 \gamma_1 ... \gamma_n = h^b \rangle. \end{aligned}$$

when $g < 0$ ([Jc], VI.9-VI.10). The element $h \in \pi_1(M)$ which occurs in these presentations is represented by any regular fibre of the Seifert structure. It generates a normal cyclic subgroup K of $\pi_1(M)$ which is central if B is orientable.

Let $\chi(B)$ be the Euler characteristic of B and recall that the orbifold Euler characteristic ([Th1], Chapter 13) of the orbifold $\mathcal{B}$ is the rational number given by

$$\chi^{orb}(\mathcal{B}) = \chi(B) - \Sigma_{i=1}^{n}(1 - \frac{1}{\alpha_i}) = \begin{cases} 2 - 2g - \Sigma_{i=1}^{n}(1 - \frac{1}{\alpha_i}) & \text{if } B \text{ is orientable} \\ 2 - g - \Sigma_{i=1}^{n}(1 - \frac{1}{\alpha_i}) & \text{if } B \text{ is non-orientable.} \end{cases}$$

The orbifold $\mathcal{B}$ is called *hyperbolic*, respectively *Euclidean*, if it admits a hyperbolic, respectively Euclidean, structure, and this condition is shown to be equivalent to the condition $\chi^{orb}(\mathcal{B}) < 0$, respectively $\chi^{orb}(\mathcal{B}) = 0$, in [Th1], Chapter 13.

Foliations will play an important role in the rest of the paper and for Seifert manifolds there is a distinguished class of such objects.

Definition: A *horizontal foliation* of a Seifert fibred manifold is a foliation of M by (possibly noncompact) surfaces which are everywhere transverse to the Seifert circles.

Though such foliations are traditionally referred to as *transverse*, we have chosen to use the equally appropriate term *horizontal* to avoid confusion with the notion of a transversely oriented foliation discussed in the next section. The combined work of various authors has resulted in a complete understanding of which Seifert bundles admit horizontal foliations. In the following theorem we consider the case where M is closed and $g = 0$.

Theorem 4.2 ([EHN], [JN2], [Na]) *Let $M = M(0; b, \frac{\beta_1}{\alpha_1}, \dots, \frac{\beta_n}{\alpha_n})$ be an orientable Seifert fibred manifold where $n \geqslant 3$, $b \in \mathbb{Z}$ and α_j, β_j are integers for which $0 < \beta_j < \alpha_j$. Then M admits a horizontal foliation if and only if one of the following conditions holds:*
(1) $-(n-2) \leqslant b \leqslant -2$.
(2) $b = -1$ *and there are relatively prime integers* $0 < a < m$ *such that for some permutation* $(\frac{a_1}{m}, \dots, \frac{a_n}{m})$ *of* $(\frac{a}{m}, \frac{m-a}{m}, \frac{1}{m} \dots, \frac{1}{m})$ *we have* $\frac{\beta_j}{\alpha_j} < \frac{a_j}{m}$ *for each* j.
(3) $b = -(n-1)$ *and after replacing each* $\frac{\beta_j}{\alpha_j}$ *by* $\frac{\alpha_j - \beta_j}{\alpha_j}$, *condition* (2) *holds.* □

5 Left-orderability and foliations

We saw in theorem 2.3 that a countable group G is left-orderable if and only if it acts effectively on $\mathbb{R}$ by order-preserving homeomorphisms. In the case of the fundamental group of a $\mathbb{P}^2$-irreducible 3-manifold M, theorem 3.2 shows that this condition can be relaxed to the existence of a homomorphism $\pi_1(M) \to Homeo_+(\mathbb{R})$ with non-trivial image. In particular if $\pi_1(M)$ acts in an orientation preserving way on $\mathbb{R}$ so that there are no global fixed points (such an action is commonly referred to as *non-trivial*), then $\pi_1(M)$ is left-orderable. Our goal in this section is to relate the existence of such an action to that of certain codimension one objects in M.

Topologists have considered left-orderability of 3-manifold groups for some time, at least implicitly. Gabai raised the problem of developing a theory of non-trivial group actions on order trees and asked some fundamental questions about the nature of those 3-manifolds whose groups admit such actions, especially those which act on $\mathbb{R}$ (§4,

[Ga2]). Roberts and Stein have shown [RS] that a necessary and sufficient condition for the fundamental group of an irreducible, non-Haken Seifert fibred manifold to act non-trivially on $\mathbb{R}$ is that the manifold admit a horizontal foliation dual to the action, a theme we shall expand on in this section and the next. Using standard techniques, it is possible to translate the existence of such actions into a topological condition. Indeed, if M is a compact, connected, orientable 3-manifold, we have just seen that a necessary and sufficient condition for $\pi_1(M)$ to be left-orderable is that there be a non-trivial action of M determined by some homomorphism $\phi\colon \pi_1(M) \to Homeo_+(\mathbb{R})$. Given such a homomorphism, one can construct (cf. remark 4.2 (ii), [Ga2]) a transversely orientable, transversely essential lamination whose order tree maps $\pi_1(M)$-equivariantly, with respect to ϕ, to $\mathbb{R}$. As it will not play any subsequent role in the paper, we direct the reader to [Ga2] for definitions and details.

One way to produce actions of a 3-manifold group $\pi_1(M)$ on the reals is by constructing $\mathbb{R}$-covered foliations. These are codimension one foliations such that the space of leaves of the pull-back foliation in the universal cover of M is $\mathbb{R}$. Many examples of hyperbolic 3-manifolds with $\mathbb{R}$-covered foliations exist. See [Fe], [Th2], and [Ca1, Ca2] for various constructions and related information. See, however, section 10 for examples of hyperbolic 3-manifolds which do not contain such foliations.

Lemma 5.1 *Let M be a compact, connected, orientable, irreducible 3-manifold and $\mathcal{F}$ a transversely oriented, $\mathbb{R}$-covered foliation in M without Reeb components. Denote by $\tilde{\mathcal{F}}$ the lift of $\mathcal{F}$ to $\tilde{M}$ and let $\phi\colon \pi_1(M) \to Homeo(\mathbb{R})$ be the homomorphism induced by the action of $\pi_1(M)$ on $\tilde{\mathcal{F}}$. Then*
(1) *the image of ϕ lies in $Homeo_+(\mathbb{R})$.*
(2) *if there exists a leaf $\tilde{\mathcal{L}}_0$ of $\tilde{\mathcal{F}}$ which is invariant under the action of $\pi_1(M)$ and whose image in M is, say, $\mathcal{L}_0$, the inclusion induces an isomorphism $\pi_1(\mathcal{L}_0) \twoheadrightarrow \pi_1(M)$. In particular M is either a solid handlebody or a product I-bundle over an orientable surface.*

Proof (1) Let $\gamma \in \pi_1(M)$ and suppose that $\phi(\gamma)$ reverses orientation of $\mathbb{R}$. Then $\phi(\gamma)$ has a unique fixed point in $\mathbb{R}$ and γ induces an orientation reversing homeomorphism of the corresponding leaf of $\tilde{\mathcal{F}}$. The image of this leaf in M would be a non-orientable surface, contrary to the fact that $\mathcal{F}$ is transversely oriented in an orientable 3-manifold.

(2) Suppose now that there is a leaf $\tilde{\mathcal{L}}_0$ of $\tilde{\mathcal{F}}$ which is invariant under the action of $\pi_1(M)$ and $\mathcal{L}_0$ is the image of $\tilde{\mathcal{L}}_0$ in M. Since the inverse image of $\mathcal{L}_0$ in $\tilde{M}$ is connected, $\pi_1(\mathcal{L}_0) \to \pi_1(M)$ is onto. But $\pi_1(\mathcal{L}_0) \to \pi_1(M)$ is injective by our hypotheses on $\mathcal{F}$. Thus $\pi_1(\mathcal{L}_0) \to \pi_1(M)$ is an isomorphism and so $\pi_1(M)$ is a surface group. If it is free, the irreducibility of M implies that it is a handlebody (cf. §5 of [He]). If it is a closed surface group, then M is a product I-bundle over $\mathcal{L}_0$ (theorem 10.2, [He]). □

Proposition 5.2 *Let M be a compact, connected, orientable, irreducible 3-manifold which admits a transversely oriented, $\mathbb{R}$-covered foliation of M without Reeb components. Then the fundamental group of M is left-orderable.*

Proof If the action is trivial (all elements act as the identity), then in particular there is a global fixed point, so that $\pi_1(M)$ is either free or a surface group (by lemma 5.1). In this case, either $\pi_1(M)$ is trivial, or $b_1(M) > 0$, so that $\pi_1(M)$ is LO (in fact O, by theorem 1.6). If, on the other hand, the action is nontrivial, then $\pi_1(M)$ is LO by theorem 3.2, because it has a representation into the left-orderable group $Homeo_+(\mathbb{R})$ with non-trivial image. □

Our goal now is to apply these considerations to Seifert fibred manifolds. The next two lemmas are well-known. We include their proofs for completeness.

Lemma 5.3 *Let M be a compact, connected, orientable Seifert fibred manifold and $\mathcal{F}$ a horizontal foliation in M. Then $\mathcal{F}$ is transversely orientable if and only if the surface underlying the base orbifold of M is orientable.*

Proof Since M is orientable, $\mathcal{F}$ is transversely orientable if and only if the Seifert circles in M can be coherently oriented, that is, if and only if M admits no vertical Klein bottles. It is easy to see that the latter condition is equivalent to the orientability of the surface underlying the base orbifold of M. □

Lemma 5.4 *Let M be a compact, connected, irreducible, orientable Seifert fibred manifold with infinite fundamental group. Let $\mathcal{F}$ be a horizontal foliation in M and $\tilde{\mathcal{F}}$ the lift of $\mathcal{F}$ to $\tilde{M}$, the universal cover of M. Then there is a homeomorphism $\tilde{M} \to \mathbb{R}^3$ which sends $\tilde{\mathcal{F}}$ to the set of planes parallel to the x-y-plane. In particular $\mathcal{F}$ is $\mathbb{R}$-covered.*

Before sketching a proof of this lemma, we derive the following consequence (cf. corollary 1.3).

Proposition 5.5 *Let M be a compact, connected, irreducible, orientable Seifert fibred manifold. Suppose the surface underlying the base orbifold of M is orientable, and M admits a horizontal foliation. If $\pi_1(M)$ is infinite, it is left-orderable.*

Proof Combining lemma 5.3 and proposition 5.2, we need only verify that $\mathcal{F}$ contains no Reeb components. But if it did, the boundary of such a component would be a horizontal torus T in M which bounds a solid torus. On the one hand, this torus would be compressible in M. On the other hand, it would lift to a horizontal plane in $\mathbb{R}^3$, by lemma 5.4, implying that it is π_1-injective. This is a contradiction. □

Proof of Lemma 5.4. Since $\pi_1(M)$ is infinite and M is irreducible, the universal orbifold cover of $\mathcal{B}$ is $\mathbb{R}^2$. Pulling back the Seifert fibration via this (orbifold branched) covering shows that there is a regular covering space $\hat{M} \to M$ where $\hat{M}$ is an S^1-bundle over $\mathbb{R}^2$. Hence $\hat{M}$ can be identified with $\mathbb{R}^2 \times S^1$ in such a way that the Seifert circles pull back to the S^1 factors, and so $\tilde{M}$ is identifiable with $\mathbb{R}^3$ in such a way that the Seifert circles pull back to the field of lines parallel to the z-axis. Note as well that if $\tau\colon \mathbb{R}^3 \to \mathbb{R}^3$ is vertical translation by 1, then τ may be taken to be a deck transformation of the universal cover $\mathbb{R}^3 \to M$. In particular $\tilde{\mathcal{F}}$ is invariant under τ. Let

$$p\colon \mathbb{R}^3 \to \mathbb{R}^2$$

be the projection onto the first two coordinates. We will show first of all that the restriction of p to any leaf of $\tilde{\mathcal{F}}$ is a homeomorphism.

Fix a leaf L of $\tilde{\mathcal{F}}$ and consider $p|L$. That $p|L$ is 1-1 follows from a classic result of Haefliger: a closed loop which is everywhere transverse to a codimension-1 foliation is not null-homotopic. (See the discussion in [Ga1], p. 611.) If there are points $(x_0, y_0, z_0), (x_0, y_0, z_1) \in L$ where $z_0 > z_1$, the vertical path between them concatenated with a path in L may be perturbed to be everywhere transverse to $\tilde{\mathcal{F}}$ (this uses the fact that $\tilde{\mathcal{F}}$ is transversely oriented). Since all loops in $\mathbb{R}^3$ are contractible, Haefliger's result shows that this is impossible. Thus $p|L$ is injective.

Surjectivity follows from the fact that $\tilde{\mathcal{F}}$ is transverse to the vertical line field and that it is invariant under τ. Firstly, transversality implies that $p(L)$ is open in $\mathbb{R}^2$. We claim that $\mathbb{R}^2 \setminus p(L)$ is open as well.

Suppose $(x_0, y_0) \in \mathbb{R}^2 \setminus p(L)$ and let $Z_0 \subset \mathbb{R}^3$ denote the vertical line through this point. For any $z \in [0, 1]$, transversality implies there is an open neighborhood $U_z \subset \mathbb{R}^3$ of (x_0, y_0, z) with the property that any leaf of $\tilde{\mathcal{F}}$ that intersects U_z will also intersect Z_0. By compactness, a finite number of such U_z will cover $(x_0, y_0) \times [0, 1]$, and one can find $\epsilon > 0$ so that $N_\epsilon(x_0, y_0) \times [0, 1]$ has the same property. Since $\tilde{\mathcal{F}}$ and Z_0 are both τ-invariant, it follows that $N_\epsilon(x_0, y_0) \subset \mathbb{R}^2 \setminus p(L)$, and we have verified that $\mathbb{R}^2 \setminus p(L)$ is open. The connectivity of $\mathbb{R}^2$ implies that $p|L$ is onto, and we've shown that $p|L$ is a homeomorphism of L onto R^2 for each leaf $L \in \tilde{\mathcal{F}}$.

It follows that each leaf of $\tilde{\mathcal{F}}$ intersects each vertical line in $\mathbb{R}^3$ exactly once and so the leaf space $\mathcal{L}(\tilde{\mathcal{F}})$ is homeomorphic to $\mathbb{R}$. Let $f\colon \mathbb{R}^3 \to \mathbb{R}$ be the composition of the map $\mathbb{R}^3 \to \mathcal{L}(\tilde{\mathcal{F}})$ with such a homeomorphism and observe that the map $p \times f\colon \mathbb{R}^3 \to \mathbb{R}^3$ defines a homeomorphism which sends $\tilde{\mathcal{F}}$ to the set of horizontal planes, which is what we set out to prove. $\square$

6 Left-orderability and Seifert fibred spaces

In this section we prove theorem 1.5. That is, the fundamental group of a compact, connected, Seifert fibred space M is left-orderable if and only if $M \cong S^3$ or one of the following holds: (1) $H_1(M; \mathbb{Z})$ is infinite and $M \not\cong \mathbb{P}^2 \times S^1$; or (2) M is orientable, $\pi_1(M)$ is infinite, the base orbifold of M is of the form $S^2(\alpha_1, \alpha_2, \ldots, \alpha_n)$, and M admits a horizontal foliation.

Throughout we take M to be a compact, connected Seifert fibred space with base orbifold $\mathcal{B}$.

First of all, note that the theorem holds for manifolds satisfying (2), (3) or (4) of proposition 4.1. Therefore we shall assume in the rest of this section that M is $\mathbb{P}^2$-irreducible and has a non-trivial fundamental group. Under these conditions, theorem 3.2 shows that $\pi_1(M)$ is LO when $b_1(M) > 0$, while proposition 5.5 shows that it is LO when $\pi_1(M)$ is infinite, the base orbifold of M is of the form $S^2(\alpha_1, \alpha_2, \ldots, \alpha_n)$, and M admits a horizontal foliation. This proves one direction of the theorem.

To prove the other direction, we assume $\pi_1(M)$ is LO. If $b_1(M) > 0$ we are done, so we assume that $b_1(M) = 0$. By lemma 3.3, M must be closed and orientable. Thus $\mathcal{B} = S^2(\alpha_1, \ldots, \alpha_n)$ or $\mathbb{P}^2(\alpha_1, \ldots, \alpha_n)$. Note as well that $\pi_1(M)$ is infinite as it is a non-trivial torsion free group. We must prove that $\mathcal{B} = S^2(\alpha_1, \ldots, \alpha_n)$ and M admits a horizontal foliation.

Our assumptions imply that $M = M(g; b, \frac{\beta_1}{\alpha_1}, \ldots, \frac{\beta_n}{\alpha_n})$ (cf. §4) where $n \geqslant 3$, $b \in \mathbb{Z}$, α_j, β_j are integers for which $0 < \beta_j < \alpha_j$, and

$$g = \begin{cases} 0 & \text{when } |\mathcal{B}| = S^2 \\ -1 & \text{when } |\mathcal{B}| = \mathbb{P}^2. \end{cases}$$

We noted in §4 that the fundamental group of M admits a presentation of the form

$$\begin{cases} \langle \gamma_1, \ldots, \gamma_n, h \mid h \text{ central }, \gamma_j^{\alpha_j} = h^{-\beta_j}, \gamma_1\gamma_2\ldots\gamma_n = h^b \rangle & \text{when } |\mathcal{B}| = S^2 \\ \\ \langle \gamma_1, \ldots, \gamma_n, y, h \mid \gamma_j^{\alpha_j} = h^{-\beta_j}, yhy^{-1} = h^{-1}, y^2\gamma_1\gamma_2\ldots\gamma_n = h^b \rangle & \text{when } |\mathcal{B}| = \mathbb{P}^2, \end{cases}$$

where $h \in \pi_1(M)$ is represented by a regular fibre. Since $\{1\} \neq \pi_1(M)$ is LO, there is a non-trivial homomorphism

$$\phi\colon \pi_1(M) \to Homeo_+(\mathbb{R}).$$

The next lemma shows that we may suppose that the action of $\pi_1(M)$ on $\mathbb{R}$ induced by ϕ has no global fixed points.

Lemma 6.1 *If there is a homomorphism $G \to Homeo_+(\mathbb{R})$ with image $\neq \{id\}$, then there is another such homomorphism which induces an action without global fixed points.*

Proof Fix a homomorphism $\phi\colon G \to Homeo_+(\mathbb{R})$ with image $\neq \{id\}$ and observe that $F := \{x \mid \phi(\gamma)(x) = x \text{ for every } \gamma \in G\}$ is a closed, proper subset of $\mathbb{R}$. Each component C of the non-empty set $\mathbb{R} \setminus F$ is homeomorphic to $\mathbb{R}$ and is invariant under the given action. Letting $f\colon \mathbb{R} \to C$ be some fixed orientation-preserving homeomorphism, we may replace each $\phi(\gamma)$ by $f^{-1}\phi(\gamma)f\colon \mathbb{R} \to \mathbb{R}$ and obtain the desired action without global fixed points. $\square$

Lemma 6.2 (compare lemma 2, [RS]) *Suppose that $M = M(g; b, \frac{\beta_1}{\alpha_1}, \ldots, \frac{\beta_n}{\alpha_n})$ where $g \in \{0, -1\}$. For a homomorphism $\phi\colon \pi_1(M) \to Homeo_+(\mathbb{R})$ the following statements are equivalent:*
(1) *The action induced by ϕ is non-trivial.*
(2) *$\phi(h)$ is conjugate to translation by 1.*

Proof It is clear that condition (2) implies condition (1), so suppose the action induced by ϕ is non-trivial. As any fixed point free element of $Homeo_+(\mathbb{R})$ is conjugate to translation by 1, we shall assume that there is some $x_0 \in \mathbb{R}$ such that $\phi(h)(x_0) = x_0$ and proceed by contradiction. Recall the presentation for $\pi_1(M)$ described above. We have $\phi(\gamma_j)^{\alpha_j}(x_0) = \phi(h)^{-\beta_j}(x_0) = x_0$ for each $j \in \{1, 2, \ldots, n\}$. As $\phi(\gamma_j)$ preserves orientation, this implies that x_0 is fixed by γ_j. In the case where $|\mathcal{B}| = \mathbb{P}^2$ we also have $\phi(y)^2(x_0) = \phi(y)^2\phi(\gamma_1\gamma_2\ldots\gamma_n)(x_0) = \phi(h)^b(x_0) = x_0$ and so $\phi(y)(x_0) = x_0$ as well. In either case x_0 is fixed by $\pi_1(M)$, contradicting the fact that the action is non-trivial. Thus $\phi(h)$ is fixed-point free and therefore is conjugate to translation by 1. □

Now we complete the proof of theorem 1.5. By lemma 6.2 we may assume that $\phi(h) \in Homeo_+(\mathbb{R})$ is translation by 1. Our assumptions on M imply that $\pi_1^{orb}(\mathcal{B}) = \pi_1(M)/\langle h \rangle$ acts properly discontinuously on $X = \mathbb{E}^2$ or $\mathbb{H}^2$. Thus ϕ induces a diagonal action of $\pi_1(M)$ on $\mathbb{R}^3 = X \times \mathbb{R}$, which can be seen to be free and properly discontinuous as $\phi(h)$ is translation by 1. Thus the quotient can be identified with M (which is a $K(\pi, 1)$). The lines $\{x\} \times \mathbb{R}$ and the planes $X \times \{t\}$ are invariant under this action and descend in M to the Seifert fibres and a horizontal foliation respectively. Further since the image of ϕ lies in $Homeo_+(\mathbb{R})$, an orientation of the vertical line field in $\mathbb{R}^3$ descends to a coherent orientation of the circle fibres in M. Thus the induced foliation is transversely orientable and so $|\mathcal{B}|$ is orientable (lemma 5.3). It follows that $\mathcal{B} = S^2(\alpha_1, \dots, \alpha_n)$. This completes the proof. □

Remark 6.3
(1) It is proved in [EHN] that there is a homomorphism $\phi\colon \pi_1(M) \to Homeo_+(\mathbb{R})$ for which $\phi(h)$ is translation by 1 if and only if M admits a transversely oriented horizontal foliation. We have already described how to construct horizontal foliations from such representations and conversely how to produce such a representation when given a horizontal foliation, at least when $b_1(M) = 0$.
(2) Lemma 6.2 does not hold when $|\mathcal{B}| \neq S^2, \mathbb{P}^2$ and this explains why the condition that $\pi_1(M)$ be left-orderable does not imply, in general, that M admits a horizontal foliation.

7 Bi-orderability and surface groups

All surface groups other than $\mathbb{Z}/2 \cong \pi_1(\mathbb{P}^2)$ are locally indicable and hence LO (cf. theorem 2.7). To see this, it suffices to observe that the cover corresponding to a given nontrivial finitely generated subgroup has infinite torsion-free homology. Our interest here focuses on the bi-orderability of these groups. We prove,

Theorem 1.6 *If S is any connected surface other than the projective plane $\mathbb{P}^2$ or Klein bottle $K = \mathbb{P}^2 \# \mathbb{P}^2$, then $\pi_1(S)$ is bi-orderable.*

The theorem is already well-known in the case of orientable surfaces: it is proved in [Ba, Lo] that their fundamental groups are residually free (and hence bi-orderable). However, the fundamental group of the non-orientable surface $S = \mathbb{P}^2 \# \mathbb{P}^2 \# \mathbb{P}^2$ is not residually free; this is because the image of any homomorphism ϕ from $\pi_1(S) = \langle a, b, c \mid a^2b^2c^2 = 1 \rangle$ to a free group is cyclic (see [LS], p.51) and therefore sends the commutator subgroup to $\{1\}$. For another approach see [GS].

In the remainder of this section we will outline a proof of this theorem. In fact, our argument fits into a larger picture, in that similar arguments have been applied to quite diverse situations - see [RoWi] (which contains further details) as well as [G-M] and [KR]. In what follows we will denote the connected sum of n projective planes by $n\mathbb{P}^2$.

We remarked above that $\pi_1(\mathbb{P}^2)$ is not LO. For $S = 2\mathbb{P}^2$, the Klein bottle,

$$\pi_1(S) \cong \langle x, y : xyx^{-1} = y^{-1} \rangle$$

is a well-known example of a group which is left-orderable (being an extension of $\mathbb{Z}$ by $\mathbb{Z}$), but not bi-orderable, as the defining relation would lead to a contradiction. If S is noncompact, or if ∂S is nonempty, then $\pi_1(S)$ is a free group, and therefore bi-orderable. Thus we are reduced to considering closed surfaces. According to the standard classification, such surfaces are either a connected sum of tori, or projective planes in the nonorientable case. The key to our analysis will be the nonorientable surface with Euler characteristic -1, namely $3\mathbb{P}^2$.

Proposition 7.1 *Let $S = 3\mathbb{P}^2$ be the connected sum of three projective planes. Then $\pi_1(S)$ is bi-orderable.*

Before proving this result, we explain how it implies theorem 1.6. Starting with the nonorientable surfaces $(n+2)\mathbb{P}^2 = T^2\#n\mathbb{P}^2$, we note that $S = 3\mathbb{P}^2 = T^2\#\mathbb{P}^2$ can be pictured as a torus with a small disk removed, and replaced by sewing in a Möbius band. Consider an n-fold cover of the torus by itself, and modify the covering by replacing one disk downstairs, and n disks upstairs, by Möbius bands. This gives a covering of S by the connected sum of T^2 with n copies of $\mathbb{P}^2$. Thus the fundamental group of $(n+2)\mathbb{P}^2$ embeds in that of $3\mathbb{P}^2$, and is therefore bi-orderable.

For the orientable surfaces S_g of genus $g \geqslant 2$ (the cases $g = 0, 1$ being easy) the result follows because S_g is the oriented double cover of $(g+1)\mathbb{P}^2$; so $\pi_1(S_g)$ is a subgroup of a bi-orderable group. This completes the proof of theorem 1.6, assuming 7.1. □

To prove proposition 7.1, our strategy is to define a surjection from $G = \pi_1(S)$ to $\mathbb{Z}^2$ with a certain kernel F, so that we have a short exact sequence

$$1 \longrightarrow F \hookrightarrow G \longrightarrow \mathbb{Z}^2 \longrightarrow 1.$$

Moreover, we shall construct a biordering on F so that the conjugation action of G on F is by order-preserving automorphisms. By lemma 2.2, this yields a biordering of G.

We recall that S is a torus with a disk removed and a Möbius band glued in its place. Squashing that Möbius band induces the desired surjection ψ: $G = \pi_1(S) \to \pi_1(T^2) = \mathbb{Z}^2$. More explicitely, G has presentation

$$G = \langle a, b, c : aba^{-1}b^{-1} = c^2 \rangle.$$

(with a and b corresponding to a free generating set of the punctured torus, and c corresponding to a core curve of the Möbius band), and ψ kills the generator c.

The kernel F, consisting of those elements with exponent sums in both a and b equal to zero, is an infinitely generated free group, with one generator for every element of $\mathbb{Z}^2$. Geometrically, we can interpret F as the fundamental group of a covering space $\widetilde{S}$

of S: starting with the universal cover $\mathbb{R}^2 \to T^2$, we remove from $\mathbb{R}^2$ a family of small disks centered at the integral lattice points, and glue in Möbius bands in their place. Thus we obtain a covering space $\widetilde{S}$ of S.

There is no canonical free generating system for F - for definiteness we may take

$$x_{i,j} = a^i b^j c b^{-j} a^{-i}.$$

So we have $F = \langle x_{i,j} \rangle; (i,j) \in \mathbb{Z}^2$.

Now G acts upon F by conjugation, which may be described in terms of the generators as follows.

Lemma 7.2 *Suppose $g \in G$ has exponent sums m and n in a and b, respectively. Then there are $w_{i,j} \in F$ such that*

$$g x_{i,j} g^{-1} = w_{i,j} x_{i+m,j+n} w_{i,j}^{-1}.$$

Proof Just take $w_{i,j} = g a^i b^{-n} a^{-i-m}$. Check exponent sums to verify $w_{i,j} \in F$. $\square$

For the following, F_{ab} denotes the abelianization of F, which is an infinitely generated free abelian group, with generators, say $\widetilde{x}_{i,j}$. Any automorphism φ of F induces a unique automorphism φ_{ab} of F_{ab}. For example, in the above lemma, conjugation by g acts under abelianization as the shift $\widetilde{x}_{i,j} \to \widetilde{x}_{i+m,j+n}$. Proposition 7.1 now follows from the

Lemma 7.3 *There is a bi-ordering of the free group $F = \langle x_{i,j} \rangle; (i,j) \in \mathbb{Z}^2$ which is invariant under every automorphism φ: $F \to F$ which induces, on F_{ab}, a uniform shift automorphism $\widetilde{x}_{i,j} \to \widetilde{x}_{i+m,j+n}$.*

Proof We use the Magnus expansion μ: $F \to \mathbb{Z}[[X_{i,j}]]$, where $\mathbb{Z}[[X_{i,j}]]$ is the ring of formal power series in the infinitely many noncommuting variables $X_{i,j}$, with the restriction that each power series may involve only finitely many variables. The Magnus map μ is given by

$$\mu(x_{i,j}) = 1 + X_{i,j}; \quad \mu(x_{i,j}^{-1}) = 1 - X_{i,j} + X_{i,j}^2 - X_{i,j}^3 + \cdots$$

Clearly the image of F lies in the group Γ of units with constant term unity, $\Gamma = \{1 + O(1)\} \subseteq \mathbb{Z}[[X_{i,j}]]$, and the image of the commutator $[F,F]$ lies in $\{1 + O(2)\}$. As done in [MKS] for the finitely-generated case, one can prove that μ: $F \to \Gamma$ is an *embedding* of groups. Elements of $\mathbb{Z}[[X_{i,j}]]$ may be written in standard form, arranged in ascending degree, and within a degree terms are arranged lexicographically by their subscripts (which in turn are ordered lexicographically). Then two series are compared by the coefficient of the "first" term at which they differ (here is where the finiteness assumption is necessary). It is well-known (see e.g. [KR]) that, restricted to Γ, this ordering is a (multiplicative) bi-ordering.

Finally, we check that the ordering is invariant under the action by φ. Since $\varphi(x_{i,j}) = x_{i+m,j+n}\, c_{i,j}$, where $c_{i,j}$ is in the commutator subgroup $[F,F]$, and since $[F,F]$ maps into $\{1 + O(2)\}$ under the Magnus embedding, we have for any $u \in F$ that the lowest nonzero-degree terms of $\mu(\varphi(u))$ coincide precisely with those of $\mu(u)$, except that all the subscripts are shifted according to the rule $X_{i,j} \to X_{i+m,j+n}$. This implies that the Magnus-ordering of F is invariant under ϕ. $\square$

8 Bi-orderability and Seifert fibred spaces

Our goal is to prove theorem 1.7: for the fundamental group of a compact, connected Seifert fibred space M to be bi-orderable, it is necessary and sufficient that it be one of $S^3, S^1 \times S^2, S^1 \tilde{\times} S^2$, a solid Klein bottle, or a locally trivial, orientable circle bundle over a surface different from $S^2, \mathbb{P}^2$ or the Klein bottle $K = 2\mathbb{P}^2$.

8.1 Sufficiency

If M is one of $S^3, S^1 \times S^2, S^1 \tilde{\times} S^2$, or a solid Klein bottle, it is clear that $\pi_1(M)$ is bi-orderable. If M is an *orientable* circle bundle over a surface $B \neq S^2, \mathbb{P}^2, K$, then $\pi_2(B)$ is trivial and the homotopy sequence of the bundle yields the exact sequence:

$$1 \to \pi_1(S^1) \to \pi_1(M) \to \pi_1(B) \to 1.$$

Since $M \to B$ is an orientable S^1-bundle, the bi-orderable group $\pi_1(S^1)$ is central in $\pi_1(M)$. Theorem 1.6 shows that $\pi_1(B)$ is bi-orderable, and therefore by lemma 2.2, $\pi_1(M)$ is bi-orderable as well.

8.2 Necessity

Throughout this subsection we use $\mathcal{B}$ to denote the base orbifold of M, B to denote the surface underlying $\mathcal{B}$, $\Sigma \subset B$ to denote the singular points of $\mathcal{B}$, and $L = \partial B \cap \Sigma$ to denote the set of reflector lines of $\mathcal{B}$. We are assuming the following:

$(*)$ M is a compact Seifert fibred 3-manifold whose fundamental group is bi-orderable.

Lemma 8.1 *If M satisfies $(*)$, the restriction of $M \to B$ to $B \setminus \Sigma$ is an orientable circle bundle. Consequently, any element of $\pi_1(M)$ represented by a regular fibre is central.* $\square$

Proof If the bundle in question were not orientable, there would be a simple closed curve C in $B \setminus \Sigma$ over which fibres could not be consistently oriented. Then M would contain a Klein bottle over C. In particular $M \ncong S^3$ and so by proposition 4.1 (2), the class $h \in \pi_1(M)$ of a regular fibre is non-trivial. If $\gamma \in \pi_1(M)$ corresponds to C, then $\gamma^{-1} h \gamma = h^{-1} \in \pi_1(M)$, and this cannot happen if $\pi_1(M)$ is biorderable. $\square$

Lemma 8.2 *In a bi-orderable group G, a non-zero power of an element γ is central if and only if γ is central.*

Proof Obviously, in any group, powers of a central element are central. On the other hand, suppose there is an integer $n > 0$ such that γ^n is central in G. If there is some $\mu \in G$ which does not commute with γ, say $\gamma\mu\gamma^{-1} < \mu$. Then by invariance under conjugation, $\gamma^2\mu\gamma^{-2} < \gamma\mu\gamma^{-1} < \mu$ and by induction $\gamma^k\mu\gamma^{-k} < \mu$ for each positive integer k. We arrive at the contradiction: $\mu = \gamma^n\mu\gamma^{-n} < \mu$. The case $\gamma\mu\gamma^{-1} > \mu$ similarly leads to a contradiction. Hence γ must be central in $\pi_1(M)$. $\square$

A case of interest arises when $G = \pi_1(M)$ and γ is a class represented by some fibre of the given Seifert structure. Evidently there is an integer $\alpha > 0$ such that γ^α is represented by a regular fibre, and therefore, as we noted above, γ^α is central.

Corollary 8.3 *Assuming M satisfies $(*)$, let $\gamma \in \pi_1(M)$ be a class represented by an arbitrary fibre of M. Then γ is central.* $\square$

We now return to the proof of the "necessity" part of theorem 1.7: that is that assuming $(*)$ we can conclude that M belongs to the given list. The proof is divided into the three cases $\Sigma = \emptyset, L \neq \emptyset$, and $\Sigma \neq L = \emptyset$.

Case 1: $\Sigma = \emptyset$.

In this case $M \to B$ is an orientable, locally trivial circle bundle (lemma 8.1). If $B \cong S^2$, then M is homeomorphic to either S^3, a lens space with non-trivial fundamental group, or $S^1 \times S^2$. Evidently the second option is incompatible with $(*)$. Suppose then that B is $\mathbb{P}^2$ or $K = 2\mathbb{P}^2$. Note that M is necessarily non-orientable. Since M satisfies $(*)$, it is clear in these cases that $M \to B$ cannot be a trivial bundle, and this fact determines M up to homeomorphism. To see this we recall that the orientable circle bundles over B are classified by the set of homotopy classes of maps $B \to BS^1$. Since $BS^1 = K(\mathbb{Z}, 2)$, these bundles correspond to elements in $H^2(B) \cong \mathbb{Z}/2$. In particular there is a unique, orientable, non-trivial circle bundle $p\colon M \to B$. In order to construct M, let D be a small 2-disk in B and set $B_0 = \overline{B \setminus D}$. Consider $M = (B_0 \times S^1) \cup_f (D^2 \times S^1)$ where $f\colon \partial B_0 \times S^1 \to S^1 \times S^1$ preserves the S^1 factors and identifies $\partial D^2 \times pt$ with a curve in $\partial B_0 \times S^1$ which wraps once around ∂B_0 and once around S^1. There is a natural map $M \to B$ which is an orientable circle bundle over B and it is simple to see that $H_1(M) \ncong H_1(B \times S^1)$. Thus this is the bundle we are looking for.

Subcase: $B = \mathbb{P}^2$. The explicit description given in the previous paragraph of the closed, connected, non-orientable manifold M shows that $\pi_1(M) \cong \mathbb{Z}$. Thus $M \cong S^1 \tilde{\times} S^2$ (by proposition 4.1 (5)), so we are done. Alternately observe that $S^1 \tilde{\times} S^2$ is a quotient of $S^1 \times S^2$ by the involution $(u, x) \mapsto (-u, -x)$ in such a way that the bundle $S^1 \times S^2 \to S^2$ quotients to a nontrivial, oriented circle bundle over $S^1 \tilde{\times} S^2 \to \mathbb{P}^2$. By the preceding paragraph, it is the only one.

Subcase: $B = 2\mathbb{P}^2$. We will show this cannot happen. As shown in §4,

$$\pi_1(M) \cong \langle x, y, t \mid t \text{ central }, t = x^2y^2\rangle \cong \langle x, y \mid x^2y^2 \text{ central}\rangle.$$

We verify x^2 is central in this group by the calculation

$$[x^2, y] = x^2yx^{-2}y^{-1} = (x^2y^2)y^{-1}x^{-2}y^{-1} = y^{-1}x^{-2}(x^2y^2)y^{-1} = 1,$$

and so by lemma 8.2, x is central as well. But this is easily seen to be false by projecting $\pi_1(M)$ onto the non-abelian group $\langle x, y \mid x^2, y^2 \rangle \cong \mathbb{Z}/2 * \mathbb{Z}/2$. We've shown that if M satisfies $(*)$, it cannot be a circle bundle over the Klein bottle.

Case 2: $L \neq \emptyset$, that is, there are reflector curves.

We will show that in this case, M is either $S^1 \tilde{\times} S^2$, a solid Klein bottle, or a trivial circle bundle over the Möbius band.

Let N be a regular neighbourhood in B of the set of reflector lines and N_0 a component of N. Let γ be the central element of $\pi_1(M)$ represented by an exceptional fibre in N_0 (cf. corollary 8.3). Set $B_0 = \overline{B \setminus N_0}$ and observe that the decomposition $B = B_0 \cup N_0$ induces a splitting $M = M_0 \cup P_0$ where $M_0 \to B_0$ and $P_0 \to N_0$ are Seifert fibrings. One readily verifies that $M_0 \cap P_0$ is a vertical torus or a vertical annulus depending on whether $L \cap N_0$ is a circle or an arc (vertical Klein bottles are ruled out by lemma 8.1). It follows that P_0 is a twisted I-bundle over a torus in the first case and a solid Klein bottle (cf. pages 433-434 of [Sc2]) otherwise. In any event, $M_0 \cap P_0$ is incompressible in P_0.

Since a fibre is never contractible in a Seifert manifold with boundary, and the only Seifert manifolds with compressible boundaries are homeomorphic to solid tori or solid Klein bottles, our assumptions imply that if $M_0 \cap P_0$ compresses in M_0, then M_0 is a solid torus, P_0 is a twisted I-bundle over the torus, and $M_0 \cap P_0 = \partial M_0$, i.e. M is a Dehn filling of P_0. Noting that a twisted I-bundle over the torus is homeomorphic to a trivial S^1-bundle over the Möbius band, it follows that $\pi_1(M)$ is a non-trivial quotient group of $\pi_1(P_0) = \mathbb{Z}^2$. On the other hand, the bi-orderability of $\pi_1(M)$ implies it has no torsion. The only possibility is for $\pi_1(M) \cong \mathbb{Z}$. Since M is closed and non-orientable, it must be $S^1 \tilde{\times} S^2$.

Assume then that $M_0 \cap P_0$ is incompressible in M_0, so that $\pi_1(M)$ is the free product of $\pi_1(M_0)$ and $\pi_1(P_0)$ amalgamated along $\pi_1(M_0 \cap P_0)$. As $\gamma \in \pi_1(P_0) \setminus \pi_1(M_0 \cap P_0)$, the only way it can be central is for $\pi_1(M_0 \cap P_0) \to \pi_1(M_0)$ to be an isomorphism. It follows that $M \cong P_0$ and so is either a solid Klein bottle or twisted I-bundle over the torus, both of which have bi-orderable fundamental groups. Noting that the latter space is homeomorphic to a trivial S^1-bundle over the Möbius band completes this part of the proof of theorem 1.7.

Case 3: $\Sigma \neq L = \emptyset$

Let the orders of the cone points in $\mathcal{B}$ be $\alpha_1, \dots \alpha_n \geqslant 2$ $(n \geq 1)$. We shall argue that M is homeomorphic to one of $S^3, S^1 \times S^2, S^1 \tilde{\times} S^2$, or the trivial bundle $D^2 \times S^1$.

Lemma 8.4 *If M satisfies $(*)$ and case 3, then there are only $n < 3$ cone points.*

Proof If $n \geqslant 3$, there are surjective homomorphisms

$$\pi_1(M) \to \pi_1^{orb}(\mathcal{B}) \to \Delta(\alpha_1, \alpha_2, \alpha_3) = \langle \bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}_3 \mid \bar{\gamma}_j^{\alpha_j} = 1, \bar{\gamma}_1\bar{\gamma}_2\bar{\gamma}_3 = 1 \rangle$$

where $\Delta(\alpha_1, \alpha_2, \alpha_3)$ is the $(\alpha_1, \alpha_2, \alpha_3)$ triangle group and $\bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}_3$ are the images of classes in $\pi_1(M)$ corresponding to the first three exceptional fibres. Since $\bar{\gamma}_1$ and $\bar{\gamma}_2$ generate the non-abelian group $\Delta(\alpha_1, \alpha_2, \alpha_3)$, γ_1 is not central in $\pi_1(M)$, contradicting lemma 8.3. $\square$

Lemma 8.5 *If M satisfies $(*)$, and Case 3, then the base orbifold has $H_1(B)$ finite, so that B is one of D^2, S^2 or $\mathbb{P}^2$. Moreover if $n = 2$, then B is D^2 or S^2.*

Proof It is not hard to see that if either $H_1(B)$ is infinite, or $n = 2$ and B is non-orientable, then there is a finite covering $f\colon \hat{B} \to B$ so that the pullback orbifold $\hat{\mathcal{B}}$ has at least three cone points. Let $M_f \to \hat{\mathcal{B}}$ be the pull-back of $M \to \mathcal{B}$, via f, so that M_f is a covering space of M, as well as a Seifert fibre space over $\hat{\mathcal{B}}$. If $\pi_1(M)$ is bi-orderable, so is $\pi_1(M_f)$, but that contradicts lemma 8.4. $\square$

Subcase: $B = \mathbb{P}^2$ and $n = 1$. Think of $\mathcal{B}$ as the union of a Möbius band without singularities and a disk containing exactly one cone point. From lemma 8.1 it follows that M is a Dehn filling of the product of a Möbius band and S^1. But then, condition $(*)$ implies that $\pi_1(M) \cong \mathbb{Z}$ and as M is closed and non-orientable, it must be homeomorphic to $S^1 \tilde{\times} S^2$ (by proposition 4.1 (5)).

Subcase: $B = S^2$ and $n = 1$ or 2. Then M is the union of two solid tori, and the only such manifolds with bi-orderable groups are S^3 and $S^1 \times S^2$.

Subcase: $B = D^2$ and $n = 1$ or 2. When $n = 2$, $\pi_1^{orb}(\mathcal{B}) \cong \mathbb{Z}/\alpha_1 * \mathbb{Z}/\alpha_2$ (as $L = \emptyset$) where the class in $\pi_1(M)$ represented by the first exceptional fibre projects to a generator of $\mathbb{Z}/\alpha_1$ under the surjection $\pi_1(M) \to \pi_1^{orb}(\mathcal{B})$. As this class is not central, this case does not arise (cf. lemma 8.3). On the other hand, if $n = 1$ then $M \cong S^1 \times D^2$. This completes the proof of the present case and hence that of theorem 1.7. $\square$

9 Orderability and Sol manifolds

The goal of this section is to investigate the orderability of the fundamental groups of Sol manifolds, and to prove theorem 1.9.

We recall from theorem 4.17 of [Sc2] that every compact, connected manifold M whose interior admits a complete Sol metric carries the structure of a 2-dimensional bundle over a 1-dimensional orbifold with a connected surface of non-negative Euler characteristic as generic fibre. When $\partial M \neq \emptyset$, this implies that M is homeomorphic to either a 3-ball, a solid torus, a solid Klein bottle, the product of a torus with an interval, or a twisted I-bundle over the Klein bottle K. Theorem 1.9 clearly holds in these cases, so from now on we shall assume that M is closed. Denoting the torus by T and the Klein bottle by K, we then have that M is either

(i) a T- or K-bundle over the circle, or

(ii) non-orientable and the union of two twisted I-bundles over K, or

(iii) orientable and the union of two twisted I-bundles over K, which are glued together along their torus boundaries.

In cases (i) and (ii), $\pi_1(M)$ is LO by theorem 3.2 and corollary 3.4.

Proposition 9.1 *Let M be a closed, connected Sol manifold.*
(1) $\pi_1(M)$ is LO if and only if cases (i) or (ii) arise, that is if and only if M is either non-orientable or orientable and a torus bundle over the circle.
(2) $\pi_1(M)$ is O if and only if M is a torus bundle over the circle whose monodromy in $GL_2(\mathbb{Z})$ has at least one positive eigenvalue.

Proof (1) It remains to prove that an orientable manifold carrying the Sol metric which is a union of two twisted I-bundles over the Klein bottle cannot have an LO fundamental group. Our proof is an adaptation of an idea of Bergman [Be2].

The Klein bottle K has fundamental group $\pi_1(K) = \langle m, l \mid l^{-1}ml = m^{-1}\rangle$ (with m and l standing for meridian and longitude respectively); any element in $\pi_1(K)$ can be written in the form $m^a l^b$ ($a, b \in \mathbb{Z}$). We note that in any left-ordering of $\pi_1(K)$ we have $m \ll |l|$, i.e. if $l^\epsilon > 1$ for some $\epsilon \in \{1, -1\}$, then $m^n < l^\epsilon$ for all $n \in \mathbb{Z}$. (For if we had $1 < l^\epsilon < m^n$, it would follow that $1 > m^{-n}l^\epsilon = l^\epsilon m^n > 1 \cdot 1 = 1$.) It follows that in any left-ordering we have $m \ll |m^a l^b|$ whenever $b \neq 0$. Observe that this condition characterizes the subgroup of $\pi_1(K)$ generated by m.

Now we recall that our 3-manifold M consists of two twisted I-bundles N_1, N_2, and $\pi_1(\partial N_i) \cong \mathbb{Z}^2$ is an index 2 subgroup of $\pi_1(N_i)$ with generators l^2 and m. With this choice of generators, the glueing map f can be described by an element of $GL_2(\mathbb{Z})$. Moreover, $\pi_1(M)$ is an amalgamated product $\pi_1(N_1) *_f \pi_1(N_2)$. Let's assume that this group is LO. By restriction, we obtain left-orderings on $\pi_1(N_1)$ and $\pi_1(N_2)$. In $\pi_1(M)$, the meridian $m_1 \in \pi_1(N_1)$ is identified with an element $f(m_1) \in \pi_1(N_2)$. By the previous paragraph, $m_1 \ll |m_1^a l_1^{2b}|$ for all $a, b \in \mathbb{Z}$ with $b \neq 0$ – note that $m_1^a l_1^{2b}$ lies in the boundary torus. Thus the same must be true for $f(m_1) \in \pi_1(N_2)$, and it follows that $f(m_1)$ is a meridian of N_2. In other words, f must glue meridian to meridian, and the 2×2-matrix representing f is of the form $\begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}$. It is well-known that there are Seifert structures on N_1 and N_2 for which m_1 and m_2 are represented by circle fibres in ∂N_1 and ∂N_2. Thus M is Seifert fibred, not Sol as hypothesized.

(2) There can be no π_1-injective Klein bottles in a manifold whose group is O, so we are reduced to the case of a torus bundle over the circle. Suppose that M is such a manifold with monodromy $A \in GL_2(\mathbb{Z})$. There is an exact sequence

$$1 \to \mathbb{Z}^2 \to \pi_1(M) \to \mathbb{Z} \to 1$$

where the right hand $\mathbb{Z}$ acts on the left-hand $\mathbb{Z}^2$ by A. Hence $\pi_1(M)$ is bi-orderable if and only if there is an bi-ordering on $\mathbb{Z}^2$ whose positive cone P is invariant under A. If we think of $\mathbb{Z}^2$ as a subgroup of $\mathbb{R}^2$, then any bi-ordering of $\mathbb{Z}^2$ is defined by a line $L \subset \mathbb{R}^2$ through the origin; the positive cone consists of the elements of $\mathbb{Z}^2$ which lie in one of the components of $\mathbb{R}^2 \setminus L$ as well as the elements of $\mathbb{Z}^2 \cap L$ which lie to one side of $0 \in L$. If one eigenvalue of A, say λ_1, is positive, with an associated eigenvector $v_1 \in \mathbb{R}^2$, there is a linearly independent eigenvector $v_2 \in \mathbb{R}^2$ for A whose associated eigenvalue λ_2 is real (since $\lambda_1\lambda_2 = \pm 1$). We claim that the positive cone P_L of the bi-order on $\mathbb{Z}^2$

defined by $L = \{tv_2 \mid t \in \mathbb{R}\}$ is invariant under the action of A. The fact that M is Sol implies that the eigenvectors of A have irrational slopes - when $|A| = 1$ this follows from the fact that $|\mathrm{trace}(A)| > 2$, and when $|A| = -1$ from the fact that $|\mathrm{trace}(A^2)| > 2$. Hence $\mathbb{Z}^2 \cap L = \{0\}$ and so P_L is the intersection of $\mathbb{Z}^2$ with a component of $\mathbb{R}^2 \setminus L$. These components are preserved by A since $\lambda_1 > 0$, and thus $A(P_L) = P_L$.

On the other hand if the eigenvalues of A are both negative, then no half-space of $\mathbb{R}^2$ is preserved by A, and therefore $\pi_1(M)$ admits no bi-ordering. □

It follows from the description of the closed, connected Sol manifolds we gave at the beginning of this section, that each such manifold is finitely covered by a torus bundle over the circle whose monodromy has positive eigenvalues. Thus,

Corollary 9.2 *The fundamental group of a closed, connected Sol manifold is virtually bi-orderable.* □

10 Hyperbolic manifolds

Finally, we consider what is perhaps the most important 3-dimensional geometry, and the least understood in terms of orderability. R. Roberts, J. Shareshian, and M. Stein have very recently discovered a family of closed hyperbolic 3-manifolds whose fundamental groups are not left-orderable. These are constructed from certain fibre bundles over S^1, with fibre a punctured torus, and pseudo-Anosov monodromy represented by the matrix $\begin{pmatrix} m & 1 \\ -1 & 0 \end{pmatrix}$, where $m < -2$ is an odd negative integer. The manifold $M^3_{p,q,m}$ is constructed by Dehn filling of this bundle, corresponding to relatively prime integers $p > q \geqslant 1$. We refer the reader to [RSS] for details of the construction. In particular, they show that

$$\pi_1(M^3_{p,q,m}) \cong \langle t, a, b : t^{-1}at = aba^{m-1}, t^{-1}bt = a^{-1}, t^{-p} = (aba^{-1}b^{-1})^q \rangle,$$

and prove that every homomorphism

$$\pi_1(M^3_{p,q,m}) \to Homeo_+(\mathbb{R})$$

is trivial (in the sense defined in section 5). It follows that $\pi_1(M^3_{p,q,m})$ is not left-orderable.

Proof of Theorem 1.10. We need to show that each of the eight geometries contains manifolds whose groups are left-orderable and others whose groups are not. For the six Seifert geometries, this is an easy consequence of theorem 1.5. First note that an S^3-manifold has an LO group if and only if it is a 3-sphere. For each of the other five Seifert geometries one can construct prime, orientable, closed manifolds with positive first Betti number and which carry the appropriate geometric structure. Such manifolds have LO groups by theorem 3.2. On the other hand, closed orientable manifolds admitting such geometries can be constructed having first Betti number 0 and non-orientable base orbifold. Theorem 1.5 implies that their groups are not LO. The case of closed manifolds admitting a Sol geometric structure can be dealt with in a similar manner. Likewise, there are many hyperbolic closed manifolds with positive first Betti number, whose groups are therefore LO. Finally, the examples of [RSS] provide many closed hyperbolic 3-manifolds with non-LO groups. □

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