An Ordering for Groups of Pure Braids and Fibre-type Hyperplane Arrangements

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1 Introduction and Statement of Results

This paper concerns the Artin braid groups B_n and especially the subgroups P_n of *pure* braids. Our main result is the explicit construction of a strict total ordering, $<$, of pure braids which is invariant under multiplication on both sides (for $\alpha, \beta, \gamma \in P_n,\ \alpha < \beta$ implies $\alpha\gamma < \beta\gamma$ and $\gamma\alpha < \gamma\beta$) and has the following properties:

1. There is a straightforward algorithm to decide which of two given pure braids is the greater. It is, in a sense, an extension of Artin's solution of the word problem for P_n.

2. The ordering respects the standard inclusions $P_n \subset P_{n+1}$, and so defines an ordering on the direct limit P_∞. It is also compatible with the homomorphism $f \colon P_{n+1} \to P_n$ which "forgets the last string," meaning that $\alpha < \beta$ implies $f(\alpha) \leq f(\beta)$.

3. The pure braids $\beta \in P_n$ which are positive in the sense of Garside [14] are also positive in our ordering. That is, if β is a nontrivial pure braid expressible in the standard braid generators σ_i with no negative exponents ($\beta \in P_n^+ = P_n \cap B_n^+$) then $1 < \beta$, where 1 denotes the identity n-string braid. (see sec. 8)

4. P_n^+ is *well-ordered* under our ordering.

5. Our ordering of P_n extends Garside's partial ordering of P_n^+. (see sec. 9)

The ordering is defined by a combination of the "combing" technique of Artin and the Magnus expansion of free groups in rings of formal power series. The existence of an ordering of P_n, invariant under multiplication on both

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sides, implies that the group ring $\mathbf{Z}P_n$ embeds in a division ring, using a result proved independently by Malćev [19] and Neumann [22][1]. An important result of Dehornoy [8] (see also [13] and [29]) is that the full braid groups B_n can be given a strict total ordering which is invariant under multiplication on one side (but not both). We will see that Dehornoy's ordering of B_n and our ordering of P_n are fundamentally incompatible; indeed any right- or left-ordering of B_n is necessarily incompatible with any bi-ordering of P_n, according to [27].

In the following sections, we describe a few basics of the braid groups, orderable groups, Artin's combing, Magnus expansions, Dehornoy's ordering and establish our particular conventions. We also discuss the connection with partial orderings which have been defined by Garside and Elrifai-Morton.

We are grateful to L. Paris for pointing out that our methods also apply to a large class of complex hyperplane arrangements, those of fibre type, as defined in [12]. Thus we can define explicit orderings to show the following:

THEOREM: *The fundamental group of the complement of a complex hyperplane arrangement of fibre type can be given a strict total ordering which is invariant under multiplication on both sides.*

As pointed out in [28], the existence of a bi-ordering of P_n (as well as the fibre-type hyperplane arrangement groups) follows directly from the observation of [12] that these groups are residually nilpotent, torsion-free (using the main result of [23]). The contribution of the present paper is the explicit construction with the properties cited above, perhaps most notably the well-ordering of Garside-positive pure braids.

2 Braid groups

Emil Artin [1] defined the braid groups B_n as isotopy classes of n disjoint strings embedded in 3-space, monotone in a given direction, and beginning and ending at certain specified points. The product of two braids is given by concatenation. These groups have played an extremely important rôle in topology, analysis, algebra and theoretical physics. Artin showed that B_n admits a presentation with generators $\sigma_1, \ldots, \sigma_{n-1}$, and relations

$$\sigma_i\sigma_j = \sigma_j\sigma_i \qquad \text{if } |i-j| > 1, \quad 1 \leq i, j \leq n-1 \tag{1}$$

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \qquad i = 1, \ldots, n-2, \tag{2}$$

where the generators σ_i can be interpreted geometrically as the so-called **elementary braids** (shown in figure 2) whose only crossing takes the $(i+1)$-st strand over the i-th; this corresponds to a right-hand screw twist.

Corresponding to any braid is a permutation of the set $\{1, \cdots, n\}$ recording how the strings connect the endpoints. In particular, a braid whose permutation is the identity is called a **pure braid**. These form a normal subgroup P_n of

[1] Added in proof: P. Linnell has announced that $\mathbf{Z}B_n$ also embeds in a division ring.

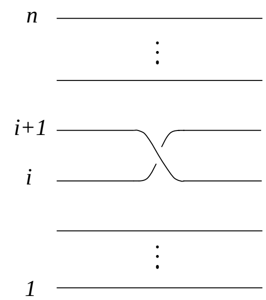


FIGURE 1: The elementary braid σ_i

index $n!$ in B_n, and fit into an exact sequence

$$1 \to P_n \to B_n \to S_n \to 1.$$

3 Automorphisms of free groups

Let F_n be a free group of rank n, with generators $x_1, \ldots, x_n$. Artin showed that B_n embeds in $\mathrm{Aut}(F_n)$, the automorphism group of the free group, in the following way. Let $\mathbf{D}_n := D^2 - Q_n$ denote the disk with n punctures $Q_n = \{p_1, \ldots, p_n\}$ arranged in order along a straight line in D^2. F_n can be identified with $\pi_1(\mathbf{D}_n)$. A braid can be regarded as the graph of a motion of n distinct points in the disk, beginning and ending in Q_n; this motion extends to the whole disk, so that a braid gives rise to a self-homeomorphism of $\mathbf{D}_n$, unique up to isotopy, fixing the punctures and boundary of D^2. Passing to the fundamental group, this induces an automorphism of F_n.

To be precise, let $x_1, \ldots, x_n$ be loops in $\mathbf{D}_n$ representing generators of $\pi_1(\mathbf{D}_n)$, oriented in the clockwise sense, as shown in Figure 2. The arcs stand for loops which are small regular neighbourhoods of the arcs.

The braid σ_i corresponds to the (say right-hand) screw motion interchanging p_i and p_{i+1} and fixed off a small neighborhood of the interval between these points. The automorphism corresponding to σ_i is given by

$$\begin{array}{rcl} x_i & \mapsto & x_i x_{i+1} {x_i}^{-1}, \\ x_{i+1} & \mapsto & x_i, \\ x_j & \mapsto & x_j \qquad (j \neq i, i+1.) \end{array}$$

This action is illustrated in Figure 2. Thus a braid β will take x_i to $w_i x_{\pi(i)} {w_i}^{-1}$, where π is the permutation corresponding to β, and w_i also depends on β. It is well-known that this determines an embedding of B_n into $\mathrm{Aut}(F_n)$. We point out for later reference that a *pure* braid corresponds to an automorphism which is a "local conjugation," i.e., one in which each generator is sent to some conjugate of itself.

Further details may be found, for example, in [5], or in [7].

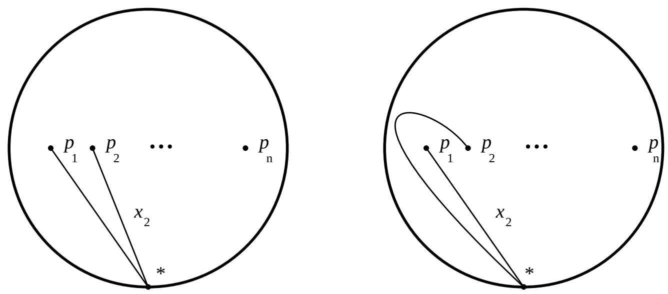


FIGURE 2: The action of σ_1 on $\pi_1(\mathbf{D}_n)$

4 Ordered groups

If G is a group whose elements can be given a strict total ordering $<$ which is right-invariant,

$$g < h \Rightarrow gk < hk,$$

for all $g, h, k \in G$, then $(G, <)$ is said to be a **right-ordered** group. If the ordering is also left-invariant, we say $(G, <)$ is a **bi-ordered** group (also called fully-ordered, or simply "ordered" in the literature.)

A right-orderable group G can be recognized by the existence of a "positive cone" $\mathcal{P} \subset G$. A group G is right-orderable if and only if there exists $\mathcal{P} \subset G$ such that: (1) $\mathcal{P}$ is multiplicatively closed: $\mathcal{P} \cdot \mathcal{P} \subset \mathcal{P}$, and (2) $G \setminus \{1\} = \mathcal{P} \coprod \mathcal{P}^{-1}$ where $\mathcal{P}^{-1} = \{p^{-1} \mid p \in \mathcal{P}\}$, i.e., $\mathcal{P}$ partitions G. For if G is a right orderable group, take $\mathcal{P} = \{g \mid 1 < g\}$. Conversely, if such a subset $\mathcal{P}$ exists, we may define a right order on G by $g < h \Leftrightarrow hg^{-1} \in \mathcal{P}$. (Note: the criterion $g^{-1}h \in \mathcal{P}$ would give a left-invariant ordering; a group is right-orderable if and only if it is left-orderable, but the orderings may differ.)

G is bi-orderable if and only if properties (1) and (2) above hold, for some subset $\mathcal{P}$ and in addition, (3): for all $g \in G$, $g\mathcal{P}g^{-1} \subset \mathcal{P}$ (normality).

It is clear that subgroups of right-orderable (resp. bi-orderable) groups are right-orderable (resp. bi-orderable) groups. The theory of orderable groups is over a century old. A classical theorem of Hölder [15] asserts that if an orderable group is Archimedean, then it is isomorphic (algebraically and in the ordering sense) with a subgroup of the additive reals, and therefore abelian. An ordering on a group is said to be **Archimedean** if whenever $1 < x < y$ in the group, there exists an integer p such that $y < x^p$.

Orderability is a strong property. For example, if G is right-orderable, then G is torsion-free. Furthermore, if R is a ring with no zero-divisors, and G is right-orderable, then the group ring RG has no zero-divisors and no units, other than the "trivial units" of the form rg where r is a unit in R and $g \in G$. It is nontrivial, but well-known (see [23], [4], [9], [16]) that free groups are bi-orderable (we will present a proof shortly). More generally, Vinogradov [30] proved that bi-orderability is preserved under free products. Good references on ordered groups are [26], [21], or [17].

Right orderability is easily seen to be preserved under extensions, that is, if $1 \to G \to K \to H \to 1$ is an exact sequence of groups and G and H are right orderable, so is K, by taking $k \in K$ to be positive iff its image in H is positive, or the image is the identity and k is positive in G. However, bi-orderability is not quite so well-behaved, being preserved under direct products, but not necessarily even semi-direct products. If G and H are bi-orderable, then $G \times H$ is also bi-orderable: simply take the lexicographic ordering: $(g, h) < (g', h') \iff g <_G g'$ or $g = g'$ and $h <_H h'$.

Now suppose G acts on H, the action being denoted by $h \mapsto h^g$. In this situation, we can define the **semi-direct product** $G \ltimes H$ to have underlying set $G \times H$, but taking as the product $(g, h)(g', h') = (gg', h^{g'}h')$. The Klein bottle group

$$\langle x, y \mid yxy^{-1} = x^{-1} \rangle$$

is not bi-orderable, although it is a semi-direct product of two infinite cyclic groups (which are, of course, bi-orderable). The reason this group cannot be bi-orderable is that x and x^{-1} are conjugate, and so one is positive if and only if the other one is positive, which leads to a contradiction. A similar argument can be made in the braid group B_3, taking $x = \sigma_1\sigma_2^{-1}$ and $y = \sigma_1\sigma_2\sigma_1$. It follows that none of the braid groups $B_n, n \geq 3$ are bi-orderable, as observed in [24].

We shall need the following lemma, whose proof is routine.

Lemma 1 *Let G and H be bi-ordered groups. Then the lexicographical order on $G \ltimes H$ is a bi-ordering if and only if the action of G on H preserves the order on H (equivalently, $(\mathcal{P}_H)^g \subset \mathcal{P}_H$ for all $g \in G$.)*

5 The Magnus ordering of the free group F_n

Let F_n denote the free group of rank n, generated by $x_1, \ldots, x_n$. By using a representation of F_n into a sufficiently large ring, due to Magnus [20], we can produce a bi-ordering with special invariance properties.

Define $\mathbf{Z}[[X_1, X_2, \ldots, X_n]]$ to be the ring of formal power series in the non-commuting variables $X_1, \ldots, X_n$, and let $\mu\colon F_n \to \mathbf{Z}[[X_1, X_2, \ldots, X_n]]$ be the **Magnus map**, defined on generators by

$$\mu\colon \begin{cases} x_i & \mapsto 1 + X_i \\ {x_i}^{-1} & \mapsto 1 - X_i + {X_i}^2 - {X_i}^3 + \cdots \end{cases}$$

Each term of a formal series has a degree, simply the sum of the exponents, and we use the usual notation $\mathcal{O}(d)$ to denote the sum of all terms of degree at least d. As noted in [20], the subset $\mathcal{G} = \{1 + \mathcal{O}(1)\}$ of $\mathbf{Z}[[X_1, X_2, \ldots, X_n]]$ is a subgroup. The inverse of an element $1 + W$ is simply $1 - W + W^2 - \cdots$, even when $W \in \mathcal{O}(1)$ is itself an infinite series. Also shown in [20] is the following.

Theorem 2 (Magnus) *The map $\mu\colon F_n \to \mathcal{G}$ is injective.*

Because of this, we may omit mention of the map μ and identify F_n with its image, writing $x_1 = 1 + X_1$, etc.

Definition 3 *We now define an ordering on* $\mathbf{Z}[[X_1, X_2, \ldots, X_n]]$. *See [20], chapter 5, exercise 5.6.10, and also [3]. First declare* $X_1 \prec X_2 \prec \cdots \prec X_n$. *If* V, W *are two distinct series, consider the smallest degree at which they differ, and sort the monomial terms of that degree with variables in lexicographic order, using* $\prec$. *Compare the coefficients of the first term, when written in the order described, at which* V *and* W *differ. Declare* $V < W$ *precisely when the coefficient of that term in* V *is smaller than the corresponding coefficient in* W. *We call this the* **Magnus ordering** *on* $\mathbf{Z}[[X_1, X_2, \ldots, X_n]]$.

Example 1 *Under the Magnus map, we have* $x_1 = 1 + X_1$ *and* $x_2 = 1 + X_2$. *The images differ in that* x_1 *has coefficient* 1 *at term* X_1 *while* x_2 *has coefficient* 0 *in this term. Hence* $x_1 > x_2$. *Similarly, we have* $x_1 > x_2 > \cdots > x_n > 1$.

Theorem 4 *The Magnus order on* $\mathbf{Z}[[X_1, X_2, \ldots, X_n]]$ *induces a bi-ordering on* $\mathcal{G}$ *and hence on* F_n. *Moreover, this ordering of* F_n *is preserved under any* $\varphi \in \mathrm{Aut}(F_n)$ *which induces the identity on* $H_1(F_n) = F_n/[F_n, F_n]$.

Proof. The ordering is clearly invariant under addition, but not necessarily under multiplication in general (for example multiplication by -1 reverses order.) However, we shall establish right-invariance under multiplication in $\mathcal{G}$ by verifying that if $u, v \in \mathcal{G}$ then we have

$$u < v \Leftrightarrow vu^{-1} > 1.$$

Write $u = 1+U$ and $v = 1+V$ where $U, V \in \mathcal{O}(1)$. It is clear from the definition that $u < v$ if and only if $V - U > 0$. In the calculation

$$vu^{-1} = (1 + V)(1 - U + U^2 - \cdots) = 1 + V - U + R$$

every term of the remainder $R = (V - U)(-U + U^2 - U^3 + \cdots)$ has degree exceeding that of the lowest degree term of $V - U$, and so $V - U$ is positive if and only if $vu^{-1} > 1$. A similar calculation shows that $u < v \Leftrightarrow u^{-1}v > 1$, and therefore the ordering is also left-invariant.

To prove the second statement, consider an automorphism $\varphi\colon F_n \to F_n$ which induces the identity on first homology. This means that $\varphi(x_i)x_i^{-1}$ is in the commutator subgroup $[F_n, F_n]$. As observed in [20], the image of $[F_n, F_n]$ lies in the subgroup $\{1 + \mathcal{O}(2)\}$ of $\mathcal{G}$. For convenience, write $x_i' = \varphi(x_i)$. The corresponding Magnus expansion is

$$x_i' = 1 + X_i' = (1 + \mathcal{O}(2))(1 + X_i) = 1 + X_i + \mathcal{O}(2)$$

and therefore

$$X_i' = X_i + \mathcal{O}(2).$$

Now if w is a word in the free group F_n, its image under φ has Magnus expansion obtained from that of w by replacing each occurence of X_i by $X_i + \mathcal{O}(2)$. It follows that the first non-zero non-constant terms of w and its image under φ are identical, and therefore φ preserves the positive cone of F_n, which is equivalent to being order-preserving. ∎

6 Artin Combing and the Structure of P_n

There is a "natural" inclusion of $P_{n-1} \hookrightarrow P_n$, which adds an n-th strand to an $(n-1)$-strand pure braid β, as illustrated in Figure 3.

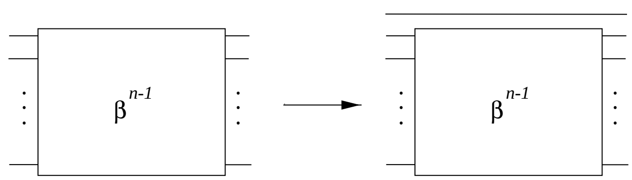


FIGURE 3: The natural inclusion of $P_{n-1} \hookrightarrow P_n$

There is also a retraction $f\colon P_n \to P_{n-1}$ defined by "deleting the n-th strand". Define F_{n-1} to be the kernel of f; then F_{n-1} consists of all braids with representatives for which strands $1, 2, \ldots, n-1$ pass straight through and the n-th strand weaves among the remaining ones. This presents $F_{n-1} \cong \pi_1(\mathbf{R}^2 \setminus (n-1)$ points), a free group of rank $n-1$, as a normal subgroup of P_n. We may write $\beta \in P_n$ as $\beta = f(\beta)f(\beta)^{-1}\beta$. Note that $f(\beta) \in P_{n-1}$ while $f(\beta)^{-1}\beta \in F_{n-1}$.

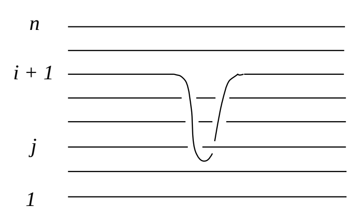


FIGURE 4: The generator $x_{j,i}$ of $F_i = \langle x_{1,i}, \ldots, x_{i,i}\rangle \subset P_n$.

For any $i \in \{1, \ldots, n-1\}$, we may thus consider the subgroup $F_i \subset P_i \subset P_n$ as consisting of all braids having all strands going straight across, except the one of index $i+1$, which may have crossings only with strings of lower index. Free generators $x_{1,i}, \ldots, x_{i,i}$ of F_i are illustrated in Figure 4. In terms of the Artin generators,

$$x_{j,i} = \sigma_i \sigma_{i-1} \cdots \sigma_{j+1} {\sigma_j}^2 {\sigma_{j+1}}^{-1} \cdots {\sigma_{i-1}}^{-1} {\sigma_i}^{-1}.$$

When the context of F_i is understood, we will drop the second subscript and write simply $F_i = \langle x_1, \ldots, x_i\rangle$.

Theorem 5 (Artin [2]) *The map $\beta \mapsto (f(\beta), f(\beta)^{-1}\beta)$ is an isomorphism $P_n \cong P_{n-1} \ltimes F_{n-1}$. The action of P_{n-1} on $F_{n-1} = \langle x_1, \ldots, x_{n-1}\rangle$ induces the identity on the abelianization $H_1(F_{n-1}) = F_{n-1}/[F_{n-1}, F_{n-1}]$.*

Proof. The semi-direct product decomposition is well-known and immediate from the observation that F_{n-1} is normal and intersects P_{n-1} in precisely the identity element. For a good explanation, see [7], which also explains that the action of P_{n-1} on F_{n-1} turns out to be the Artin action of B_{n-1} as described in Section 3, restricted to the pure braids. As remarked earlier, each such automorphism is a local conjugation, and therefore the identity upon abelianization. ∎

The full "Artin combing" of P_n is the iterated semi-direct product decomposition $P_n \cong (\cdots(F_1 \ltimes F_2) \ltimes F_3) \ltimes \cdots \ltimes F_{n-1})$. This provides a unique factorization of a pure braid β as a product $\beta = \beta_1\beta_2\ldots\beta_{n-1}$ in which the pure braid β_i belongs to F_i. Thus we introduce coordinates in the semi-direct product:

$$\beta = (\beta_1, \beta_2, \ldots, \beta_{n-1}),$$

in which β_i may be expressed in terms of the generators $x_1, \ldots, x_i$ of F_i.

Theorem 6 *The lexicographic order on $P_n \cong F_1 \ltimes F_2 \ltimes F_3 \ltimes \cdots \ltimes F_{n-1}$, with terms in the free factors compared using the Magnus order, is a bi-ordering.*

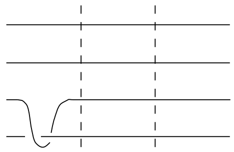
Proof. This is just a recursive application of Theorem 5, Theorem 4 and Lemma 1. ∎

We will refer to this order on P_n as the **Artin-Magnus** order.

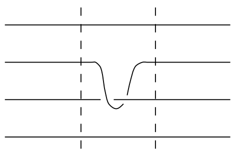
7 Examples and properties of the ordering

We will begin with some examples in $P_4 = (F_1 \ltimes F_2) \ltimes F_3$. An element $\beta \in P_4$ will be written $\beta = (\beta_1, \beta_2, \beta_3)$, where each $\beta_i \in F_i$. The convention for labeling generators of the F_i is explained in Figure 4.

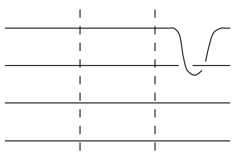
Example 2 *First consider the simplest nontrivial pure braids.*



$${\sigma_1}^2 = (x_1, 1, 1) = (1 + X_1, 1, 1).$$



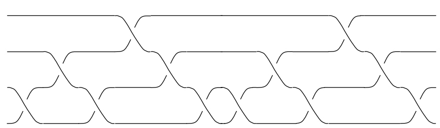
$${\sigma_2}^2 = (1, x_2, 1) = (1, 1 + X_2, 1).$$



$${\sigma_3}^2 = (1, 1, x_3) = (1, 1, 1 + X_3).$$

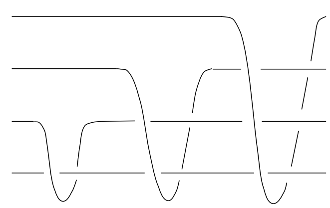
This shows that ${\sigma_1}^2 > {\sigma_2}^2 > {\sigma_3}^2 > 1$. In general, ${\sigma_i}^2 > {\sigma_{i+1}}^{2N}$ for every power N, illustrating that for $n > 2$, the Artin-Magnus ordering on P_n is non-Archimedean (as it must be, by Hölder's theorem).

Example 3 *The generator of the center of P_4:*



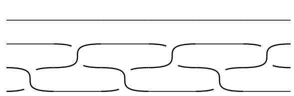
Δ^2

$=$



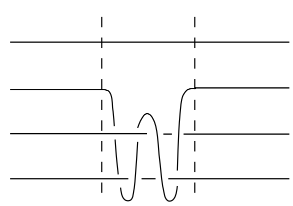
$(x_1, x_1x_2, x_1x_2x_3)$

Example 4 *The common braid used for hair, on the first three strings.*



$(\sigma_1{\sigma_2}^{-1})^3$

$=$



$(1, {x_2}^{-1}{x_1}^{-1}x_2x_1, 1)$

Applying the Magnus expansion to the second coordinate,

$$\begin{aligned}
&{x_2}^{-1}{x_1}^{-1}x_2x_1 \mapsto \\
&\quad (1 - X_2 + {X_2}^2 - {X_2}^3 + \cdots)(1 - X_1 + {X_1}^2 - {X_1}^3 + \cdots)(1 + X_2)(1 + X_1) \\
&\quad\quad = 1 - X_1X_2 + X_2X_1 + \mathcal{O}(3)
\end{aligned}$$

Since the first non-constant term is negative, we conclude $(\sigma_1{\sigma_2}^{-1})^3 < 1$.

The Artin-Magnus ordering is natural in several senses. Consider the increasing sequence

$$P_1 \subset P_2 \subset \cdots \subset P_n \subset \cdots \subset P_\infty,$$

where the group P_∞ is the direct limit.

Proposition 7 *If $m < n$, the Artin-Magnus ordering of P_n, restricted to P_m, is the Artin-Magnus ordering of P_m. Therefore, this defines an ordering of P_∞. Moreover, the retraction map $f\colon P_n \to P_{n-1}$ is as order-preserving as possible: $\beta < \gamma \Longrightarrow f(\beta) \le f(\gamma)$.*

Proof. The statement is obvious when considering coordinates: P_m extends to P_n by adding an appropriate number of "identity" coordinates. The retraction simply deletes the last coordinate. ∎

It will be useful to consider the linking numbers associated with a pure braid β. Define the **linking number** of the i-th strand with the j-th strand, $i \neq j$ (indices refer to the left-endpoints), by

$$\mathrm{lk}_{i,j}(\beta) = \frac{1}{2}\sum_c \mathrm{sign}(c),$$

where the sum is over all crossings c involving the two strings of index i and j, and sign(c) is the power, ± 1, of the corresponding braid generator ${\sigma_k}^{\pm 1}$. Each $\mathrm{lk}_{i,j}\colon P_n \to \mathbf{Z},\ 1 \le i \neq j \le n$ is additive: $\mathrm{lk}_{i,j}(\alpha\beta) = \mathrm{lk}_{i,j}(\alpha) + \mathrm{lk}_{i,j}(\beta)$.

Proposition 8 *Let β be a pure braid with n strands, expressed in Artin coordinates: $\beta = (\beta_1, \ldots, \beta_{n-1})$. Then the Magnus expansion*

$$\beta_i = 1 + q_1X_1 + \cdots + q_iX_i + \mathcal{O}(2)$$

in $F_i \subset \mathbf{Z}[[X_1, X_2, \ldots, X_i]]$ has coefficients of its linear terms:

$$q_j = \mathrm{lk}_{j,i+1}(\beta).$$

Proof. The formula is a straightforward calculation, using two observations. First, the exponent sum of $x_j = x_{j,i}$ in a word $w = w(x_1, \ldots, x_i)$ equals the coefficient of X_j in the Magnus expansion of w. Second, if $j \le i$, then $x_{j,i}$ contributes $+1$ to $\mathrm{lk}_{j,i+1}(\beta) = \mathrm{lk}_{i+1,j}(\beta)$ and zero to all other linking numbers. ∎

The **abelianization** functor takes all the braid groups P_n and $B_n,\ n \geq 2$, to infinite cyclic groups. The abelianization map $B_n \to \mathbf{Z}$ can be taken to be the total exponent count of a braid when expressed in the generators σ_i. This map sends P_n to $2\mathbf{Z}$; in this setting it can also be interpreted as twice the total linking number

$$\beta \to 2 \sum_{1 \leq i < j \leq n} \mathrm{lk}_{i,j}(\beta).$$

The abelianization map on P_n is clearly not order-preserving. However, it does respect the orderings if one confines attention to the free subgroups $F_i \subset P$ in the Artin decomposition

$$P_n = F_1 \ltimes F_2 \ltimes F_3 \ltimes \cdots \ltimes F_{n-1}.$$

The abelianization of F_i is the free abelian group of rank i, which we may identify with i-tuples of integers, $\mathbf{Z}^i$, and order lexicographically. In this setting the abelianization may be realized as the map $\mathrm{ab}_i(w) = (q_1, \ldots, q_i)$ where q_j is the exponent sum of x_j in w. We shall say that a mapping $x \to x'$ of ordered sets is **order-respecting** if $x < y$ implies $x' \leq y'$. Then we see, from the definition of the Magnus ordering of F_i that ab_i is order respecting. The next proposition follows directly from the above discussion.

Proposition 9 *With the Artin-Magnus ordering on P_n and lexicographic ordering on the product of infinite cyclic groups, the product of abelianization maps* $\mathrm{ab}_1 \times \mathrm{ab}_2 \times \cdots \times \mathrm{ab}_{n-1}$*, as a mapping*

$$P_n = F_1 \ltimes F_2 \ltimes \cdots \ltimes F_{n-1} \to \mathbf{Z} \times \mathbf{Z}^2 \times \cdots \times \mathbf{Z}^{n-1},$$

is order-respecting. Moreover, it is compatible with the retraction $f \colon P_n \to P_{n-1}$ in that the following diagram commutes:

$$\begin{array}{ccc} P_n & \xrightarrow{\mathrm{ab}_1 \times \cdots \times \mathrm{ab}_{n-1}} & \mathbf{Z} \times \mathbf{Z}^2 \times \cdots \times \mathbf{Z}^{n-1} \\ \downarrow f & & \downarrow \text{projection} \\ P_{n-1} & \xrightarrow{\mathrm{ab}_1 \times \cdots \times \mathrm{ab}_{n-2}} & \mathbf{Z} \times \mathbf{Z}^2 \times \cdots \times \mathbf{Z}^{n-2} \end{array}$$

8 Garside Positive Braids

In his celebrated solution of the conjugacy problem in B_n, Garside [14] introduced the so-called "positive" braids, i.e., nontrivial braids which have an expression in the generators σ_i without any negative exponents. We will call such braids "Garside positive." Let B_n^+ denote the semigroup of Garside positive braids together with the identity.

Theorem 10 *If a braid in P_n is Garside positive, then it is also positive in the Artin-Magnus ordering.*

Proof. The statement is clear for elements of P_2. Inductively, suppose that it is true for P_{n-1}. Let $\beta \in P_n$ be Garside positive, and consider the braid $f(\beta) \in P_{n-1}$. Note that $f(\beta)$ is either Garside positive (case (1)) or has no crossings (case (2)). In case (1), applying induction, $f(\beta)$ is Artin-Magnus positive, hence β is positive, by definition of the lexicographic order of $P_{n-1} \ltimes F_{n-1}$. In case (2), β is an element of F_{n-1}. We may assume the first $n-1$ strings are straight and read the expression for β in terms of the free generators of F_{n-1} from the places where the n^{th} string passes under the first $n-1$ strings. By our convention (choice of $x_{j,i}$), and Garside positivity, β is therefore a strictly positive word in $x_1, \ldots, x_n$. It follows from Proposition 8 that its Magnus expansion has positive leading coefficient (occurring at a linear term). Therefore $\beta > 1$. ∎

Recall that an ordered set is said to be **well-ordered** if every non-empty subset has a least element.

Theorem 11 *The set of Garside positive pure n-braids is well-ordered by the Artin-Magnus ordering.*

Proof. The statement is clear for $n = 2$. Now suppose that $n > 2$ and the theorem holds for $n-1$. Consider a non-empty set $S \subset P_n$ of Garside positive braids. We need to show S has a least element. The set $f(S) = \{f(\beta) \in P_{n-1} \mid \beta \in S\}$, being a subset of $P_{n-1} \cap B^+_{n-1}$, has a least element; call it α_0. Let

$$S_0 = \{\gamma \in S \mid f(\gamma) = \alpha_0\};$$

clearly if there is a least element of S_0 it is also the least element of S. Note that the coordinates of $\gamma \in S_0$ in $P_{n-1} \ltimes F_{n-1}$ are $\gamma = (\alpha_0, {\alpha_0}^{-1}\gamma)$. We now appeal to lemma 12 to find a $\gamma_0 \in S_0$ whose last coordinate ${\alpha_0}^{-1}\gamma_0$ is minimal. Then γ_0 is the least element of S. ∎

Lemma 12 *Consider a subset $T \subset F_{n-1} \subset P_n$ of the form*

$$T = \{{\alpha_0}^{-1}\gamma \mid \gamma \in S_0\},$$

where S_0 is some set of Garside-positive pure n-braids and α_0 is a fixed pure braid in $P_{n-1} \subset P_n$. Then T has a least element in the Magnus ordering of F_{n-1}.

Proof. The condition ${\alpha_0}^{-1}\gamma \in F_{n-1}$ is equivalent to the equation $f(\gamma) = \alpha_0$. Since α_0 is in P_{n-1}, and γ is Garside positive, we have $\mathrm{lk}_{j,n}({\alpha_0}^{-1}\gamma) = \mathrm{lk}_{j,n}(\gamma) \geq 0$. This implies that the coefficients $(q_1, \ldots, q_{n-1})$ of the linear terms in the Magnus expansion of all elements of T are positive, hence there is a lexicographically minimal one, say $(\hat{q}_1, \ldots, \hat{q}_{n-1})$. Let

$$T' = \{\tau \in T \mid \tau = 1 + \hat{q}_1 X_1 + \cdots \hat{q}_{n-1} X_{n-1} + \mathcal{O}(2)\}.$$

Then T' is nonempty, and its elements are $<$ other elements of T. We now claim that T' is finite. It follows that T' has a least element, which is also the least element of T. To verify the claim, observe that the exponent sum (as a word in $x_1, \ldots, x_{n-1}$) of every element of T' is $\hat{q}_1 + \cdots + \hat{q}_{n-1}$. Each x_i has exponent sum $+2$ when expanded in the braid generators σ_j. The σ-exponent sum is an invariant of braids, and we use it to conclude that for every ${\alpha_0}^{-1}\gamma \in T'$, the length of γ is exactly $2\sum_i \hat{q}_i$ minus the exponent sum of α_0. There are only finitely many distinct γ satisfying this. ∎

We note that the analogue of Theorem 11 was proved in [6, 18] for the Dehornoy ordering of the full braid groups.

We next compare the Artin-Magnus ordering $<_{AM}$ on P_n with Dehornoy's and several other orderings on the braid groups appearing in the literature.

9 The Garside partial order

Garside [14] defined a partial order $\prec$ on the semigroup B_n^+ by $\alpha \prec \beta$ for $\alpha, \beta \in B_n^+$, if there exists a $\gamma \in B_n^+$ such that $\alpha\gamma = \beta$. Thurston [11] showed that this in fact defines a lattice order on the set of non-repeating braids $D = \{\alpha \in B_n^+ \mid \alpha \prec \Delta\}$.

Proposition 13 *The Artin-Magnus order on P_n extends the Garside order on $P_n^+ = P_n \cap B_n^+$.*

Proof. This follows immediately from Theorem 10. ∎

10 The partial order of Elrifai and Morton

In [10], Elrifai and Morton define a partial order on B_n, as follows: for $\alpha, \beta \in B_n$, write $\alpha \leq_{EM} \beta$ when $\beta = \gamma_1\alpha\gamma_2$, for some $\gamma_1, \gamma_2 \in B_n^+$. Then $1 \leq_{EM} \alpha$ if and only if $\alpha \in B_n^+$. Furthermore, taking $\Delta = \Delta_n \in B_n$ to be the "half-twist" braid, defined inductively by $\Delta_2 = \sigma_1$, $\Delta_n = \Delta_{n-1}\sigma_{n-1}\cdots\sigma_1$, they show that in their partial order each generator σ_i satisfies

$$1 \leq_{EM} \sigma_i \leq_{EM} \Delta. \tag{3}$$

We remark that $1 \leq_{EM} {\sigma_i}^2 \leq_{EM} \Delta^2$ holds also for each generator σ_i, since $\Delta = \gamma_1\sigma_i = \sigma_i\gamma_2$, for some $\gamma_1, \gamma_2 \in B_n^+$. Hence $\Delta^2 = \gamma_1{\sigma_i}^2\gamma_2$. This is consistent with the Artin-Magnus order on P_n:

Proposition 14 *For each generator $\sigma_i \in B_n$,*

$$1 <_{AM} {\sigma_i}^2 \leq_{AM} \Delta^2.$$

We have equality in the second inequality only when $i = 1, n = 2$.

Proof. The inequalities follow immediately from example 3. ∎

However, the following example shows that the Artin-Magnus order on P_n does *not* extend the partial order defined by Elrifai and Morton. Let $\alpha = {\sigma_1}^2{\sigma_2}^{-2}$. This has Artin-combing $(x_1, {x_2}^{-1})$, hence is positive in the Artin-Magnus ordering. Taking $\gamma_1 = \sigma_1\sigma_2$ and $\gamma_2 = \sigma_2\sigma_1$, and defining $\beta = \gamma_1\alpha\gamma_2$, we have $\beta <_{AM} 1$, so in particular, $\beta < \alpha$ in the Artin-Magnus order. We note that $\gamma_1\alpha{\gamma_1}^{-1}$ is also negative in the Artin-Magnus ordering — conjugation by non-pure braids is not order-preserving.

11 The Dehornoy ordering

The Dehornoy ordering [8] on B_n may be defined in terms of the generators $\sigma_1, \cdots, \sigma_{n-1}$ as follows: A braid β is in the positive cone if and only if there is, for some $i \in \{1, \dots, n-1\}$, an expression

$$\beta = w_1\sigma_i w_2\sigma_i \cdots w_{k-1}\sigma_i w_k \tag{4}$$

in which each w_j is a word in $\sigma_{i+1}^{\pm 1}, \cdots, \sigma_{n-1}^{\pm 1}$. In other words, the generator with the lowest subscript appears with only positive exponent. A more geometric view of the same ordering appears in [13]. Then we define a right-ordering by $\alpha < \gamma$ iff $\gamma\alpha^{-1}$ is in the positive cone. (Actually Dehornoy chose his ordering to be left-invariant by using the criterion $\alpha^{-1}\gamma$ positive, but the choice is arbitrary and right-invariance seems to dominate the literature on ordered groups.)

As noted already, no ordering of B_n can be a bi-ordering, for $n \geq 3$. Even when restricted to P_n, as noted in [13], the Dehornoy ordering is not bi-invariant. However, there are some similarities between the Dehornoy and Artin-Magnus orderings. In both orderings we have

$${\sigma_1}^2 \gg {\sigma_2}^2 \gg {\sigma_3}^2 \gg \cdots > 1,$$

where the notation $\alpha \gg \beta$ means that α is greater than all powers of β. Another similarity is that the Dehornoy ordering is also a well-ordering when applied to Garside positive braids (see [18] and [6]).

On the other hand, the pure braid $(\sigma_1{\sigma_2}^{-1})^3$ is clearly Dehornoy positive, whereas we saw that it is negative in the Artin-Magnus ordering. Of course, its inverse is Dehornoy negative and Artin-Magnus positive.

We wish to thank R. Fenn for pointing out the following proposition. It shows that the Dehornoy ordering is fundamentally different from our ordering of P_n.

Theorem 15 *B_n has a least positive element in the Dehornoy ordering, namely σ_{n-1}. Similarly, ${\sigma_{n-1}}^{-1}$ is the greatest element which is < 1 in B_n. In P_n, ${\sigma_{n-1}}^2$ is the least Dehornoy-positive, and ${\sigma_{n-1}}^{-2}$ is the greatest element which is < 1 in the Dehornoy ordering.*

Proof. If a braid has the form of eq. (4), call it i-positive, so that $\beta \in B_n$ is Dehornoy-positive if and only if it is i-positive for some $i = 1, \ldots, n-1$. To prove the first statement, note that σ_{n-1} is $(n-1)$-positive. Suppose that there is a $\beta \in B_n$ with $1 < \beta < \sigma_{n-1}$. Then $\beta {\sigma_{n-1}}^{-1} < 1$ by right-invariance. Now β is i-positive for some i. If $i < n-1$ we conclude $\beta {\sigma_{n-1}}^{-1}$ is also i-positive, contradicting $\beta {\sigma_{n-1}}^{-1} < 1$. On the other hand, if $i = n-1$, β must be a positive power of σ_{n-1}, contradicting $\beta < \sigma_{n-1}$. This establishes the first statement. The other parts follow similarly. ∎

Corollary 16 *The Dehornoy ordering of B_n is discrete: every element β has a unique successor, $\sigma_{n-1}\beta$, and predecessor, ${\sigma_{n-1}}^{-1}\beta$. Similarly, in the Dehornoy ordering restricted to P_n, a pure braid β is the only element of P_n strictly between ${\sigma_{n-1}}^{-2}\beta$ and ${\sigma_{n-1}}^{2}\beta$.*

Theorem 17 *For $n \geq 3$, the Artin-Magnus ordering of P_n is order-dense: given any two pure n-braids $\alpha < \gamma$, there exist (infinitely many) pure braids β with $\alpha < \beta < \gamma$.*

Proof. By invariance, we may assume $\alpha = 1$. In the free group $F_{n-1} = \langle x_1, \ldots, x_{n-1} \rangle$, consider the sequence of commutators $\{c_i\}$ defined recursively by

$$c_1 = x_1 x_2 {x_1}^{-1} {x_2}^{-1}, \quad c_k = x_1 c_{k-1} {x_1}^{-1} {c_{k-1}}^{-1}.$$

A simple calculation shows that their Magnus expansions are, in ascending order:

$$c_k = 1 + {X_1}^k X_2 - k {X_1}^{k-1} X_2 X_1 + \cdots .$$

It follows that each $c_k > 1$ in the Magnus order of F_{n-1}, but that $c_k \to 1$ in the sense that if $1 < w \in F_{n-1}$, then $1 < c_k < w$ for sufficiently large k; just take k bigger than the degree of the first nonzero non-constant term of the Magnus expansion of w. Now, in P_n, consider the sequence of braids $\{\beta(k)\}$ with Artin coordinates

$$\beta(k) = (1, \ldots, 1, c_k).$$

It is clear that given any $\gamma > 1$ in P_n, we have $1 < \beta(k) < \gamma$ for all sufficiently large k. ∎

Since for $n \geq 3$, P_n with the Artin-Magnus ordering is a countable totally ordered set, dense and with no greatest or least element, it is order-isomorphic with the rational numbers, by a classical result. Of course this order-isomorphism cannot be an algebraic isomorphism, as P_n is non-abelian.

We note that for B_∞ or P_∞ the Dehornoy ordering is no longer discrete, but is (like the Artin-Magnus ordering) order-dense.

Theorem 18 *For $n \geq 3$, the Artin-Magnus ordering on P_n does not extend to any right-ordering on B_n. It also does not extend to a left-invariant ordering on B_n.*

Proof. We may assume that $n = 3$. It has already been shown that $(\sigma_1 {\sigma_2}^{-1})^3 < 1$. A similar calculation shows that $({\sigma_1}^{-1}\sigma_2)^3 < 1$. If the ordering extends to a right- (or left-) invariant ordering of B_3, we must have $\sigma_1{\sigma_2}^{-1} < 1$ and ${\sigma_1}^{-1}\sigma_2 < 1$.

Assuming right invariance we conclude

$$\sigma_1{\sigma_2}^{-1}{\sigma_1}^{-1}\sigma_2 < {\sigma_1}^{-1}\sigma_2 < 1,$$

which would imply $(\sigma_1{\sigma_2}^{-1}{\sigma_1}^{-1}\sigma_2)^3 < 1$. But a direct calculation shows

$$(\sigma_1{\sigma_2}^{-1}{\sigma_1}^{-1}\sigma_2)^3 = (1, {x_2}^{-1}{x_1}^{-1}{x_2}^{-1}x_1{x_2}^2) = (1, 1 + X_1X_2 + \cdots) > 1,$$

a contradiction showing no right-invariant extension to B_3 exists.

If instead we assume left-invariance, we argue that

$$\sigma_1{\sigma_2}^{-1}{\sigma_1}^{-1}\sigma_2 < \sigma_1{\sigma_2}^{-1} < 1,$$

and reach the same contradiction. ∎

Recently, Rhemtulla and Rolfsen [27] have shown that no right- or left-invariant ordering of B_n whatsoever can be bi-invariant when restricted to P_n, or any finite index subgroup.

12 Fibre-type hyperplane arrangements

A hyperplane arrangement is a collection of complex $(n-1)$-dimensional hyperplanes in complex n-space. The theory of hyperplane arrangements is a deep subject with many unsolved problems. The fundamental group of the complement of the union of the hyperplanes is an important invariant of an arrangement (see the recent survey paper by Paris [25]). It is conjectured, but not known in general, that all such groups are torsion-free.

The pure braid group P_n can be viewed as the fundamental group of the complement of the hyperplanes $z_i = z_j$, $1 \le i < j \le n$ in $\mathbf{C}^n$. Fibre-type hyperplane arrangements are defined in [12], and are in a sense a generalization of this example.

Theorem 19 *Let G be the fundamental group of (the complement of) a fibre-type hyperplane arrangement. Then G is bi-orderable.*

Proof. By definition of a fibre-type arrangement, its complement M_r is the top of a tower of fibrations $M_i \to M_{i-1}$, with fibre the complex plane minus d_i points. The space M_1 at the bottom of the tower is likewise a punctured complex plane. Therefore $\pi_1(M_i) \cong \pi_1(M_{i-1}) \ltimes F_{d_i}$. Moreover, according to [12], proposition 2.5, the action of $\pi_1(M_{i-1})$ upon F_{d_i} is trivial on $H_1(F_{d_i})$. It follows, in the same way as Theorem 6, that if each free group F_{d_i} is given the Magnus ordering, then the lexicographic ordering on

$$G \cong (\cdots(F_{d_1} \ltimes F_{d_2}) \ltimes F_{d_3}) \ltimes \cdots \ltimes F_{d_r})$$

is bi-invariant. ∎

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