Theorem 1. If f has an antiderivative in a neighbourhood of a curve α, say $F' = f$, then

$$\int_\alpha f(z)\mathrm{d}z = F(z_f) - F(z_i)$$

where z_f, resp. z_i, is the final, resp. initial, point of α.
Corollary: In the situation above but with a closed curve

$$\oint_\alpha f(z)\mathrm{d}z = 0.$$

Theorem 2 (Cauchy). Let α be a closed curve. If f is holomorphic in a neighbourhood of the interior of α, then

$$\oint_\alpha f(z)\mathrm{d}z = 0.$$

Discussion. The case $\frac{1}{z^n}$ and a closed curve winding around $z = 0$.

If $n = 2, 3, \ldots$, then $\frac{1}{z^n}$ has an antiderivative in a neighbourhood of the unit circle, namely $\frac{1}{1-n}\frac{1}{z^{n-1}}$, and by Theorem 1,

$$\oint_\alpha \frac{1}{z^n}\mathrm{d}z = 0, \qquad (n = 2, 3, \ldots).$$

Theorem 2 does not tell us anything in this case because $\frac{1}{z^n}$ is not holomorphic at $z = 0$.

If $n = 1$, then $\frac{1}{z}$ does not have an antiderivative in a neighbourhood of the unit circle. We do have

$$(\mathrm{Log}(z))' = \frac{1}{z}, \qquad z \in \mathbb{C} \setminus (-\infty, 0],$$

but the branch cut is met along the path. Neither Theorem 1 nor Theorem 2 is of any direct help.

Finally: If the closed curve is any other curve that does not contain the singularity $z = 0$, then Theorem 2 applies for $n = 1, 2, 3 \ldots$ and yields

$$\oint_\alpha f(z)\mathrm{d}z = 0.$$

Theorem 3 (Cauchy's integral formula). Under the assumptions of Theorem 2,

$$f(w) = \frac{1}{2\pi\mathrm{i}}\oint_\alpha \frac{f(z)}{z-w}\mathrm{d}z$$

for any w in the interior of α.
Corollary: The mean-value property:

$$f(w) = \frac{1}{2\pi}\int_0^{2\pi} f(w + r\mathrm{e}^{\mathrm{i}t})\mathrm{d}t$$

whenever f is holomorphic in a neighbourhood of the disk $|z - w| \le r$.

We now prove Cauchy's theorem. This goes in two steps. We first consider a triangle T with boundary τ, a function f that is holomorphic in a neighbourhood of the triangle, and prove that

$$I = \oint_{\tau} f(z)\mathrm{d}z = 0.$$

Secondly, we use this result to construct an antiderivative F for a function f that is holomorphic in a disk, and conclude by the corollary of Theorem 1.

Step 1. We decompose T into four smaller triangles by picking the middle of each of the three sides of the triangle, see the figure. Since contributions of the integrals along the inner line segments cancel out pairwise, we see that I is the sum of the integrals along each of the four smaller triangles. We pick the one integral that is largest in absolute value and call that triangle T_1 and its boundary τ_1. Hence,

$$|I| \leq 4\,|I_1|\,, \quad \text{where} \quad I_1 = \oint_{\tau_1} f(z)\mathrm{d}z.$$

Note that the length ℓ_1 of τ_1 is half of the length ℓ of τ. The procedure can be repeated with T_1 instead of T, yielding $|I_1| \leq 4\,|I_2|$ and $\ell_2 = (1/2)\ell_1$, hence $|I| \leq 4^2\,|I_2|$ and $\ell_2 = (1/2^2)\ell$. Iterating n times, we conclude that

$$|I| \leq 4^n\,|I_n|\,, \qquad \ell_n = 2^{-n}\ell.$$

There is one point, call it z_0 that belongs to all of the triangles T_n. The function f being differentiable, we have by definition that

$$r(z, z_0) = \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \longrightarrow 0 \tag{1}$$

as $z \to z_0$. Reorganizing this,

$$f(z) = f(z_0) + (z - z_0)f'(z_0) + (z - z_0)r(z, z_0).$$

Now,

$$\oint_{\tau_n} f(z_0)\mathrm{d}z = 0, \qquad \oint_{\tau_n} (z - z_0)f'(z_0)\mathrm{d}z = 0,$$

since τ_n is a closed curve and both the constant function $z \mapsto f(z_0)$ and $z \mapsto (z - z_0)f'(z_0)$ have antiderivatives. It follows that

$$\begin{aligned} |I_n| = \left|\oint_{\tau_n} f(z)\mathrm{d}z\right| &= \left|\oint_{\tau_n} (z - z_0)r(z, z_0)\mathrm{d}z\right| \\ &\leq \ell_n \max\{|z - z_0|\,|r(z, z_0)| : z \in \tau_n\}. \end{aligned}$$

Since z_0 belongs to the interior of the triangle, $|z - z_0| \leq \ell_n$ for any z that lies somewhere on its boundary. Furthermore, if we denote by M_n the maximal value of $|r(z, z_0)|$ along τ_n, the observation (1) implies that $M_n \to 0$. Putting all together, we have

$$|I| \leq 4^n\,|I_n| \leq 4^n \frac{\ell}{2^n}\frac{\ell}{2^n} M_n = \ell^2 M_n \longrightarrow 0$$

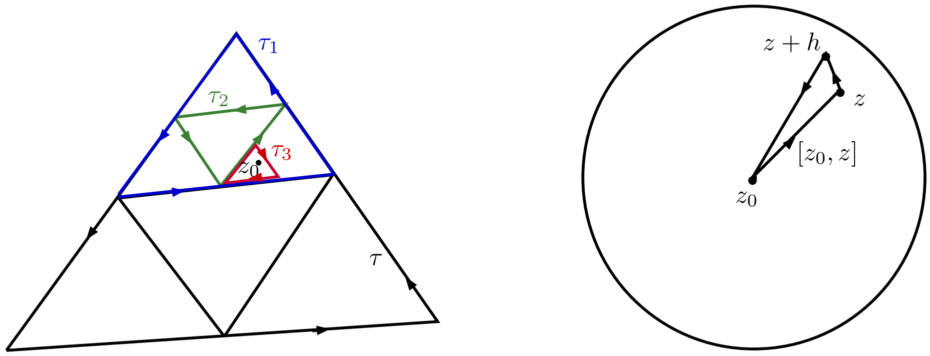

Figure 1: The sequence of triangles used in Step 1 on the left. The geometric setting of Step 2 on the right.

as $n \to \infty$. Hence $I = 0$, which is what we had set to prove.

Step 2. Let the disk Ω be centred at z_0 with radius $r > 0$. For any $z \in \Omega$, we define

$$F(z) = \int_{[z_0,z]} f(w)\mathrm{d}w$$

where $[z_0, z]$ is the line segment from z_0 to z (which lies completely in the disk), and we claim that F is an antiderivative of f. Any point close to z can be represented by $z+h$ with $|h|$ small. The three points $z_0, z, z+h$ define a triangle, see the figure again, in which f is holomorphic. By the definition of F and Step 1,

$$F(z) + \int_{[z,z+h]} f(z)\mathrm{d}z - F(z+h) = 0. \tag{2}$$

Since f is holomorphic, it is uniformly continuous and so

$$\frac{1}{h}\int_{[z,z+h]} f(z)\mathrm{d}z = \frac{1}{h}\int_0^1 f(z+ht)h\mathrm{d}t \longrightarrow f(z)$$

as $h \to 0$. With this, (2) yields

$$\frac{F(z+h) - F(z)}{h} = \frac{1}{h}\int_{[z,z+h]} f(z)\mathrm{d}z \longrightarrow f(z), \qquad (h \to 0)$$

establishing the claim and Cauchy's theorem.