1. A strong version of *Taylor's theorem.* Cauchy's integral formula for a holomorphic $f : \Omega \to \mathbb{C}$ yields

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j$$

where

$$a_j = a_j(f, z_0) = \frac{f^{(j)}(z_0)}{j!}$$

and the series is uniformly convergent on any open disk that is contained in Ω.

Consequence: If z_0 is a zero of f, then

$$f(z) = (z - z_0)^n g(z)$$

for some $n \in \mathbb{N}$ called the *order* of z_0 and a holomorphic function g. Since zeros are isolated, there is a small disk D centered at z_0 in which $g \neq 0$.

2. Now z_0 is a *pole* of order n of f if it is a zero of order n of $1/f$. If z_0 is a pole of order n of f, then

$$f(z) = (z - z_0)^{-n} h(z)$$

where h is holomorphic in D. In other words, if z_0 is a pole of f, then

$$f(z) = \sum_{j=-n}^{\infty} a_j (z - z_0)^j$$

and one defines the *residue* of f at z_0 by

$$\mathrm{Res}(f; z_0) = a_{-1}$$

It follows that

$$\mathrm{Res}(f; z_0) = \lim_{z \to z_0} \frac{1}{(n-1)!} \frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}} \big((z - z_0)^n f(z) \big)$$

whenever z_0 is a pole of order n

The Residue Theorem. If f is holomorphic in $\Omega \setminus \{z_1, \dots z_N\}$, then

$$\oint_\alpha f(z) \mathrm{d}z = 2\pi \mathrm{i} \sum_i \mathrm{Res}(f; z_i)$$

for any closed, positively oriented, simple curve in $\Omega \setminus \{z_1, \dots, z_N\}$, where the sum extends to all poles contained in the interior of α.

Remarks. (i) The order of a zero z_0 is the order of the first non-vanishing derivative at z_0
(ii) If g is holomorphic in Ω, then $f(z) = g(z)/(z - z_0)$ has a simple pole at z_0 with

$$\mathrm{Res}(f; z_0) = g(z_0)$$

which recovers Cauchy's integral formula.
(iii) A function that is holomorphic in $\Omega \setminus \{z_1, \dots, z_N\}$ is called *meromorphic* in Ω.