Some tests for the convergence or divergence of series

1. Divergence Test:
If $(a_n)_{n=1}^{\infty}$ does not converge to zero, then $\sum_{n=1}^{\infty} a_n$ diverges.
Note: If $(a_n)_{n=1}^{\infty}$ converges to zero, then the test does not give any information.

2. Comparison Test:

   1. If $\sum_{n=1}^{\infty} b_n$ converges and $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$, then $\sum_{n=1}^{\infty} a_n$ converges.
   2. If $\sum_{n=1}^{\infty} b_n$ diverges and $0 \leq b_n \leq a_n$ for all $n \in \mathbb{N}$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Note: 'for all $n \in \mathbb{N}$' can be replaced by 'for all $n \geq N_0$'.

3. Limit Comparison Test:
Let $(b_n)_{n=1}^{\infty}$ be a sequence s.t. $b_n > 0$ for all $n \in \mathbb{N}$. Let $(a_n)_{n=1}^{\infty}$ be another sequence. Assume that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L$$

exists. If $L \neq 0$, then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.
Note: In the case $L = 0$, we only have that if $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges.

4. Ratio Test:
Let $(a_n)_{n=1}^{\infty}$ be s.t. $a_n \neq 0$ for all $n \geq N_0$. Assume that

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$$

exists. The sum $\sum_{n=1}^{\infty} a_n$ converges if $L < 1$ and diverges if $L > 1$.
Note: The test does not give any information in the case $L = 1$.

5. Alternating Series Test:
Let $(a_n)_{n=1}^{\infty}$ be s.t. $a_n \geq 0$ for all $n \in \mathbb{N}$. If

$$a_{n+1} \leq a_n \qquad \text{and} \qquad \lim_{n\to\infty} a_n = 0$$

Then the series $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ converges.