MATH 421/510, 2019WT2

Homework set 1 – due January 17

Problem 1. Let $(S, \mathcal{T})$ be a topological space. Let $X \subset S$. Define

$$\mathcal{T}_X = \{O \cap X : O \in \mathcal{T}\}.$$

(i) Prove that $(X, \mathcal{T}_X)$ is a topological space ($\mathcal{T}_X$ is the *relative topology* induced by $\mathcal{T}$).
(ii) Assume that X is open. Prove that $A \subset X$ is open in X if and only if it is open in S.
(iii) Assume that X is closed. Prove that $A \subset X$ is closed in X if and only if it is closed in S.

Problem 2. Let M be a metric space. For any $y \in M, r > 0$, let $B_r(y) = \{z \in M : d(y, z) < r\}$.
(i) Prove that the set $\mathcal{T}$ of $X \subset M$ such that $x \in X$ implies $B_r(x) \subset X$ for some $r > 0$, together with $\emptyset$, is a topology.
(ii) Prove that M is first countable.
(iii) Prove that M is second countable if and only if it is separable.
(iv) Prove that M is Hausdorff.

Problem 3. Let S be an uncountable set of points. Let

$$\mathcal{T} = \{Y \subset S : S \setminus Y \text{ is finite }\} \cup \{\emptyset\}$$

(i) Prove that $\mathcal{T}$ is a topology.
(ii) Prove that $X \cap Y \neq \emptyset$ for any nonempty $X, Y \in \mathcal{T}$ (in particular, $(S, \mathcal{T})$ is not Hausdorff).
(iii) Prove that there is no countable base at any $x \in S$.
(iv) Prove that $x = \lim_{n\to\infty} x_n$ for all $x \in S$ and all sequences $(x_n)_{n\in\mathbb{N}}$ in S obeying $x_n \neq x_m$ for all $n \neq m$.

Problem 4. (i) Let S be a first countable topological space. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in S. We say that x is a *cluster point* of $(x_n)_{n\in\mathbb{N}}$ if for every neighbourhood N_x of x, $x_n \in N_x$ for infinitely many n. Prove that x is a cluster point of $(x_n)_{n\in\mathbb{N}}$ iff there is a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ that converges to x.
(ii) Let $S = [0, 1]$, and let the non-empty sets of $\mathcal{T}$ be defined by: $X \in \mathcal{T}$ if X^c is countable.
(a) Prove that $\overline{[0, 1)} = S$
(b) Prove that there is no sequence $(x_n)_{n\in\mathbb{N}}$ in $[0, 1)$ such that $\lim_{n\to\infty} x_n = 1$. Conclude that $(S, \mathcal{T})$ is not first countable.

Problem 5. A topological space S is called *disconnected* if there exist nonempty open sets U, V such that $U \cap V = \emptyset$ and $U \cup V = S$. Otherwise S is called *connected*. A subset of S is called connected/disconnected if it is connected/disconnected as a topological space with the relative topology inherited from S.
(i) Prove that S is connected if and only if the only subsets of S that are both open and closed are $\emptyset$ and S.
(ii) Prove that if $\{E_\alpha\}_{\alpha\in I}$ is a collection of connected subsets of S such that $\bigcap_{\alpha\in I} E_\alpha \neq \emptyset$, then $\bigcup_{\alpha\in I} E_\alpha$ is connected.
(iii) Prove that if $X \subset S$ is connected, then $\overline{X}$ is connected.
(iv) Prove that every point $x \in S$ is contained in a unique maximal connected subset of S and that this subset is closed. It is called the connected component of x.