MATH 421/510, 2019WT2

Homework set 10 – due March 27

Problem 1. Let V be a Banach space and let $T \in \mathcal{L}(V)$. Let $\Omega \supset \sigma(T)$ and let $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ be analytic. Let γ be a positively oriented simple closed C^2-path in $\Omega \cap \rho(T)$ whose interior contains $\sigma(T)$. Define

$$f(T) = \frac{1}{2\pi \mathrm{i}} \oint_\gamma (z1 - T)^{-1} f(z) dz.$$

(i) Let $P(z) = \sum_{j=1}^N a_j z^j$ be a polynomial. Prove that $P(T) = \sum_{j=1}^N a_j T^j$

Hint. See equation (2) in Solution 9

(ii) Prove that the *(holomorphic) functional calculus* $f \mapsto f(T)$ is a homomorphism between the algebra of functions analytic in Ω into the algebra $\mathcal{L}(V)$.

Hint. Express $(z1 - T)^{-1}(w1 - T)^{-1}$ as a difference. You are allowed to commute integrals without justification. Recall that if $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ is analytic, then $(2\pi\pi)^{-1} \oint_\gamma \frac{f(z)}{z-w} dz$ equals $f(w)$, respectively 0, if w is inside, respectively outside, of the contour γ.

(iii) Show that $\sigma(f(T)) = f(\sigma(T))$.

Problem 2. (i) Let $\|v\| = \langle v, v\rangle^{1/2}$ be a Hilbert space norm. Prove that it obeys the parallelogram law

$$\|u + v\|^2 + \|u - v\|^2 = 2\|u\|^2 + 2\|v\|^2$$

(ii) Let V be a complex vector space. Let $\|\cdot\|$ be a norm on V obeying the parallelogram law. Prove that

$$\langle u, v\rangle = \frac{1}{4}\left(\|u + v\|^2 - \|u - v\|^2 - \mathrm{i}\|u + \mathrm{i}v\|^2 + \mathrm{i}\|u - \mathrm{i}v\|^2\right)$$

is an inner product.

Hint. That $\langle v, u\rangle = \overline{\langle u, v\rangle}$, that $\langle u, \mathrm{i}v\rangle = \mathrm{i}\langle u, v\rangle$, and that $\langle u, v + w\rangle = \langle u, v\rangle + \langle u, w\rangle$ is a result of simple but tedious calculations, which do not need to be provided. In order to prove $\langle u, \lambda v\rangle = \lambda\langle u, v\rangle$, for any $\lambda \in \mathbb{C}$, prove it first for λ having rational real and imaginary parts, then prove the Cauchy-Schwarz inequality, and use it to extend to $\lambda \in \mathbb{C}$.

Problem 3.

Let $\mathcal{H}$ be a Hilbert space and let $A \in \mathcal{L}(\mathcal{H})$. The *adjoint* of A is the operator A^* defined by $\langle v, A^*w\rangle = \langle Av, w\rangle$ for all $v, w \in \mathcal{H}$.

(i) Prove that $\|A\| = \|A^*\|$.

(ii) Prove that $\sigma(A^*) = \{\overline{\lambda} : \lambda \in \sigma(A)\}$ and that if A is invertible, then $\sigma(A^{-1}) = \{\lambda^{-1} : \lambda \in \sigma(A)\}$.

(iii) Prove that if A is *self-adjoint*, namely $A = A^*$, then

$$\|A\| = \sup\{|\langle v, Av\rangle|/\|v\|^2 : v \in \mathcal{H}\}$$

(iv) Prove that if A is *normal*, namely $AA^* = A^*A$ then $r(A) = \|A\|$. *Hint.* Show $\|A\|^2 = \|AA^*\|$.

(v) Prove that for any $B \in \mathcal{L}(\mathcal{H})$, $\sigma(AB) \cup \{0\} = \sigma(BA) \cup \{0\}$.

Hint. Compute $(\lambda 1 - AB)(1 + A(\lambda 1 - BA)^{-1}B)$

Problem 4. Let $(X, \mu), (Y, \nu)$ be two measure spaces and let k be a measurable function on $X \times Y$ such that

$$\int_{X \times Y} |k(x,y)|^2 d\mu(x) d\nu(y) < \infty.$$

Prove that $K : L^2(Y, \nu) \to L^2(X, \mu)$ defined by

$$(Kf)(x) = \int_Y k(x,y) f(y) d\nu(y)$$

is such that for any bounded sequence $(f_n)_{n \in \mathbb{N}}$, the sequence $(Kf_n)_{n \in \mathbb{N}}$ has a convergent subsequence. Such an operator is called *compact*.