MATH 421/510, 2019WT2

Homework set 11 – due April 03

Problem 1. Let $A : \mathbb{N} \times \mathbb{N} \to \mathbb{C}$ obey

$$\sup_{n\in\mathbb{N}} \sum_{m\in\mathbb{N}} |A(n,m)| < \infty \qquad \text{and} \qquad \sup_{m\in\mathbb{N}} \sum_{n\in\mathbb{N}} |A(n,m)| < \infty.$$

Prove that $(Ab)_n = \sum_{m\in\mathbb{N}} A(n,m)b_m$ defines a bounded linear operator on ℓ^2 and give a bound on its norm.

Problem 2. Let $\mathcal{H} = L^2(X,\mu)$. Assume that for any measurable $A \subset X$ with $\mu(A) = \infty$, there is $B \subset A$ such that $0 < \mu(B) < \infty$. Let $f : X \to \mathbb{C}$ be a bounded measurable function. Let T_f be the operator on $\mathcal{H}$ defined by $(T_f g)(x) = f(x)g(x)$. *Hint for both (i,ii).* Use characteristic functions.
(i) Prove that $\lambda \in \sigma(T_f)$ iff

$$\mu\{x \in X : |f(x) - \lambda| < \epsilon\} > 0.$$

for all $\epsilon > 0$.
(ii) Prove that λ is an *eigenvalue* of T_f, namely there is $g_\lambda \in \mathcal{H}$ such that $T_f g_\lambda = \lambda g_\lambda$ iff

$$\mu\{x \in X : f(x) = \lambda\} > 0.$$

(iii) Determine the spectrum and the set of eigenvalues in the case $X = (0,1)$ with Lebesgue measure and $f(x) = x$.

Problem 3. Let $f(\theta) \in L^2([0,2\pi])$. For $n \in \mathbb{Z}$, let $\varphi_n(\theta) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{\mathrm{i}n\theta}$ and $c_n = \langle \varphi_n, f\rangle$.
(i) Prove that $\lim_{n\to\infty} c_{\pm n} = 0$.
(ii) Let $S_N(\theta) = \sum_{n=-N}^{N} c_n \varphi_n(\theta)$. Extend f to a periodic function on $\mathbb{R}$ and prove that

$$S_N(\theta) = \frac{1}{2\pi} \int_0^{2\pi} f(x+\theta) \frac{\sin(N+1/2)x}{\sin\frac{x}{2}}\, dx$$

Problem 4. Assume that f is periodic of period 2π and continuously differentiable. Define φ_n, c_n and S_N as in Problem 2.
(i) Prove that $\lim_{N\to\infty} S_N(\theta) = f(\theta)$. *Hint.* Prove that the integral of the kernel in S_N equals 1.
(ii) Let $b_n = \langle \varphi_n, f'\rangle$; prove that $\sum_{n=-N}^{N} |b_n|^2$ and $\sum_{n=-N}^{N} n^2 |c_n|^2$ are convergent as $N \to \infty$.
(iii) Prove that $\sum_{n=-N}^{N} |c_n|$ is convergent as $N \to \infty$.
(iv) Prove that $S_N(\theta)$ converges uniformly to $f(\theta)$.