MATH 421/510, 2019WT2

Homework set 3 – due January 31

Problem 1. Let X be a metric space with metric d. For any nonempty set $E \subset X$, define

$$d_E : X \to [0, \infty), \qquad d_E(x) = \inf\{d(x, y) : y \in E\}.$$

(i) Show that d_E is uniformly continuous on X.
(ii) Let A, B be disjoint nonempty closed subsets of X. Examine the relevance of the function

$$f(x) = \frac{d_A(x)}{d_A(x) + d_B(x)}$$

to Urysohn's lemma.

Problem 2. Let (M, d) be a metric space and let $K \subset M$ be compact.
(i) Let $\mathcal{C}$ be an open cover of K. Prove that there is $r > 0$ such that for any $x \in K$, there is $O_x \in \mathcal{C}$ with $B_r(x) \subset O_x$.
(ii) Let U be open and such that $K \subset U$. Prove that there is $r > 0$ such that $B_r(x) \subset U$ for all $x \in K$.

Problem 3. Let $P_0 = 0$ and define the sequence of polynomials $(P_n)_{n \in \mathbb{N}}$ by

$$P_{n+1}(x) = P_n(x) + \frac{1}{2}(x^2 - (P_n(x))^2) \qquad (n \geq 0).$$

Prove that $\lim_{n\to\infty} P_n(x) = |x|$ uniformly in $[-1, 1]$.

Problem 4. Let $(S, \mathcal{T})$ be a topological space and let $f : S \to \mathbb{R}$. The function f is called *lower semicontinuous* if $\{x : f(x) > a\}$ is open for any $a \in \mathbb{R}$. It is called *upper semicontinuous* if $\{x : f(x) < a\}$ is open for any $a \in \mathbb{R}$.
(i) Prove that f is continuous if and only if it is both upper and lower semicontinuous.
(ii) Let O be open. Prove that the characteristic function χ_O is lower semicontinuous.
(iii) Let C be closed. Prove that the characteristic function χ_C is upper semicontinuous.
(iv) Let $\{f_\alpha : \alpha \in I\}$ be a family of lower semincontinuous functions, and let $f = \sup\{f_\alpha : \alpha \in I\}$. Prove that f is lower semincontinuous.

Problem 5. Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ be compact Hausdorff spaces.
(i) The family $\{O_X \times O_Y : O_X \in \mathcal{T}_X, O_Y \in \mathcal{T}_Y\}$ is a base for the *box topology* $\mathcal{T}$ on $X \times Y$. Prove that $(X \times Y, \mathcal{T})$ is compact.
(ii) Let $f \in C_{\mathbb{R}}(X \times Y)$ and let $\epsilon > 0$. Prove that there is $n \in \mathbb{N}$ and functions $\{g_j \in C_{\mathbb{R}}(X) : 1 \leq j \leq n\}$ and $\{h_j \in C_{\mathbb{R}}(Y) : 1 \leq j \leq n\}$ such that

$$\left| f(x, y) - \sum_{j=1}^{n} g_j(x) h_j(y) \right| < \epsilon$$

for all $x \in X, y \in Y$.