MATH 421/510, 2019WT2

Homework set 4 – due February 07

Problem 1. Let (M,d) be a compact metric space. Prove that $\|f\| = \sup\{|f(x)| : x \in M\}$ is a well-defined norm on $C_{\mathbb{C}}(M)$ with respect to which $C_{\mathbb{C}}(M)$ is complete.

Problem 2. Let $(V, \|\cdot\|)$ be a normed linear space.
(i) Assume that V is complete. Prove that $W \subset V$ is complete iff W is closed.
(ii) Let $\{x_1, \dots, x_n\}$ be a linearly independent set in V. Prove that there is $c > 0$ such that

$$\Big\| \sum_{i=1}^{n} \lambda_i x_i \Big\| \geq c \sum_{i=1}^{n} |\lambda_i|.$$

(iii) Assume that W is finite dimensional. Prove that W is complete.

Problem 3. Let V be a vector space. Two norms $\|\cdot\|_1, \|\cdot\|_2$ are *equivalent* if there are positive constants c, C such that $c\|v\|_2 \leq \|v\|_1 \leq C\|v\|_2$ for all $v \in V$.
(i) Prove that equivalent norms generate the same topology.
(ii) Let V be finite dimensional. Prove that all norms are equivalent.

Problem 4. Let V be a normed linear space and let $C \subset V$ be a proper closed subspace. We say that $v \sim w \Leftrightarrow v - w \in C$, denote $[v] = \{w \in V : w \sim v\}$ and let $V/C = \{[v] : v \in V\}$ be the set of equivalence classes.
(i) Prove that $\|[v]\| = \inf\{\|w\| : w \in [v]\}$ is a norm on V/C.
(ii) Assume that V is complete. Prove that V/C is complete.

Problem 5. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space.
(i) Let $1 \leq p < q \leq \infty$. Prove that if $f \in L^p(\Omega) \cap L^q(\Omega)$ then $f \in L^r(\Omega)$ for all $p \leq r \leq q$.
(ii) Let $p < \infty$. Prove that if $f \in L^p(\Omega) \cap L^\infty(\Omega)$ then $\|f\|_\infty = \lim_{q\to\infty} \|f\|_q$.