MATH 421/510, 2019WT2

Homework set 7 – due March 06

Problem 1. *This is an optional exercise. It will not be graded.*
Let V be a real normed vector space, and let A, B be non empty, disjoint and convex subsets of V. Assume that A is open.
(i) Let $a_0 \in A, b_0 \in B$ and $x_0 = b_0 - a_0$. Let $C = A - B + x_0 = \{a - b + x_0 : a \in A, b \in B\}$. Prove that C is convex, open and $0 \in C, x_0 \notin C$.
(ii) Define the *Minkowski functional* as the map $p : V \to \mathbb{R}$

$$p(x) = \inf\{\lambda > 0 : x \in \lambda C\}.$$

Prove that there is $M > 0$ such that $p(x) \leq M\|x\|$ and that $C \subset \{x \in V : p(x) < 1\}$.
(iii) Prove that p is convex. *Hint.* Show first that $p(\alpha x) = \alpha p(x)$ for $\alpha > 0$.
(iv) Prove that there is $\ell \in V^*$ and $\lambda \in \mathbb{R}$ such that

$$\ell(a) < \lambda \leq \ell(b)$$

for all $a \in A, b \in B$. *Hint.* Let $f : \text{span}\{x_0\} \to \mathbb{R}$ be defined by $f(tx_0) = t$. Use Hahn-Banach.
Assume now that A is compact and B is closed.
(v) Prove that there is $\ell \in V^*$ and $\lambda \in \mathbb{R}$ such that

$$\sup\{\ell(a) : a \in A\} < \lambda < \inf\{\ell(b) : b \in B\}.$$

In other words, the convex sets A, B can be separated by the hyperplane $\{x \in V : \ell(x) = \lambda\}$.

Problem 2. Let V be a vector space and let $\|\cdot\|_1, \|\cdot\|_2$ be two norms on V such that $\|v\|_1 \leq c\|v\|_2$ for all $v \in V$. Prove that if V is complete with respect to both norms, then they are equivalent.

Problem 3. Let V, W be two Banach spaces with norms $\|\cdot\|_V, \|\cdot\|_W$. Let $T \in \mathcal{L}(V, W)$ be such that $\text{Ran}(T)$ is closed and $\dim\text{Ker}(T) < \infty$. Let $\|\cdot\|$ denote another norm on V such that $\|x\| \leq M\|x\|_V$ for all $x \in V$. Prove that there exists $C > 0$ such that

$$\|x\|_V \leq C(\|Tx\|_W + \|x\|)$$

for all $x \in V$. *Hint.* Argue by contradiction.

Problem 4. Let $V = \{z \in \ell^1 : \sum_{n=1}^{\infty} n|z_n| < \infty\}$.
(i) Prove that V is a proper dense subspace of ℓ^1
(ii) Let $T : V \to \ell^1$ be defined by $(Tz)_n = nz_n$. Prove that T is unbounded and closed.
(iii) Prove that $S = T^{-1} : \ell^1 \to V$ is bounded and surjective but not open.

Problem 5. Let V, W be Banach spaces and let $D \subset V$ be a dense subspace. Let $T : D \to W$ be a bounded linear transformation. Prove that there is a unique extension $\tilde{T} : V \to W$ such that $\|\tilde{T}\|_{\mathcal{L}(V,W)} = \|T\|_{\mathcal{L}(D,W)}$.

Problem 6. Let V be an infinite-dimensional normed linear space.
(i) Let $\ell_1, \ldots, \ell_n \in V^*$. Prove that there is $v_0 \in V, v_0 \neq 0$ such that $\ell_j(v_0) = 0$ for $1 \leq j \leq n$.
(ii) Show that the weak closure of the unit sphere $\{v \in V : \|v\| = 1\}$ is the closed unit ball $\{v \in V : \|v\| \leq 1\}$.
(iii) Show that the open unit ball $\{v \in V : \|v\| < 1\}$ is not weakly open.