MATH 421/510, 2019WT2

Homework set 9 – due March 20

Problem 1. Let V be a Banach space and Ω an open subset of $\mathbb{C}$.
A function $f : \Omega \to V$ is called *analytic* at z_0 if the following limit exists:

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

It is called *weakly analytic* if the function $F_\ell : \Omega \to \mathbb{C}$ given by $F_\ell(z) = \ell(f(z))$ is analytic at z_0 for any $\ell \in V^*$.
(i) Prove that f is analytic in Ω if and only if f is weakly analytic in Ω.
Hint. Show that $(f(z_n) - f(z_0))/(z_n - z_0)$ is Cauchy. Use Cauchy's integral formula for $F_\ell(z)$.
(ii) Prove that if f is analytic in Ω and $K \subset \Omega$ is compact, then $\|f(z)\|$ is bounded on K.
(iii) Let f be analytic in Ω. Let $w \in \Omega$ and let γ be a positively oriented simple closed C^2-path in Ω whose interior contains w. Show that

$$f(w) = \frac{1}{2\pi \mathrm{i}} \oint_\gamma \frac{f(z)}{z - w} dz.$$

Hint. Use ordinary complex analysis. You are allowed to commute integrals and linear functionals without justification.
(iv) Let f be analytic in Ω, let $z_0 \in \Omega$ and let $r > 0$ be such that $B_r(z_0) \subset \Omega$. Prove that f has a unique power series representation

$$f(z) = \sum_{n=0}^{\infty} A_n (z - z_0)^n, \qquad A_n \in \mathcal{L}(V),$$

where the series is convergent as a norm limit of its partial sums.

Problem 2. Let V be a Banach space and let $T \in \mathcal{L}(V)$. The *resolvent set* of T is defined as

$$\rho(T) = \{\lambda \in \mathbb{C} : \lambda 1 - T \text{ is invertible}\}$$

while the *spectrum* is $\sigma(T) = \mathbb{C} \setminus \rho(T)$.
(i) Prove that $\rho(T)$ is open
(ii) Prove that the function $\lambda \mapsto (\lambda 1 - T)^{-1} \in \mathcal{L}(V)$ is analytic on $\rho(T)$.
(iii) Let $\lambda \in \rho(T)$. Prove that

$$\|(\lambda 1 - T)^{-1}\| \geq d^{-1}$$

where $d = \mathrm{dist}(\lambda, \sigma(T))$.
(iv) Prove that $\sigma(T)$ is closed, bounded and nonempty.
Hint. Use the geometric series of $(\lambda 1 - T)^{-1}$ for large λ. Compute $\oint \zeta^k (\zeta 1 - T)^{-1} d\zeta$.
(v) Prove that $r(T) = \max\{|\lambda| : \lambda \in \sigma(T)\}$.

Problem 3. *This exercise will be part of HW 10. Its solution does not need to be submitted with the other two problems.*
Let V be a Banach space and let $T \in \mathcal{L}(V)$. Let $\Omega \supset \sigma(T)$ and let f be analytic in Ω. Let γ be a positively oriented simple closed C^2-path in $\Omega \cap \rho(T)$ whose interior contains $\sigma(T)$. Define

$$f(T) = \frac{1}{2\pi \mathrm{i}} \oint_\gamma (z1 - T)^{-1} f(z) dz.$$

(i) Let $P(z) = \sum_{j=1}^N a_j z^j$ be a polynomial. Prove that $P(T) = \sum_{j=1}^N a_j T^j$.
(ii) Prove that the *(holomorphic) functional calculus* $f \mapsto f(T)$ is a homomorphism between the algebra of functions analytic in Ω into the algebra $\mathcal{L}(V)$.

Hint. Express $(z1 - T)^{-1}(w1 - T)^{-1}$ as a difference. You are allowed to commute integrals without justification.

(iii) Show that $\sigma(f(T)) = f(\sigma(T))$.