MATH 421/510, 2019WT2

Homework set 1 – Solution

Problem 1. (i) By definition $O \cap X \subset X$ so that $\mathcal{T}_X \subset \mathcal{P}(X)$. $\emptyset \in \mathcal{T}$ implies $\emptyset \in \mathcal{T}_X$, and $S \in \mathcal{T}$ implies $X = S \cap X \in \mathcal{T}_X$. The finite intersection property holds in $\mathcal{T}_X$ since $\cap_{j=1}^n (O_j \cap X) = (\cap_{j=1}^n O_j) \cap X$ and the finite intersection property in $\mathcal{T}$. The arbitrary union property follows similarly from $\cup_{\alpha\in I}(O_\alpha \cap X) = (\cup_{\alpha\in I} O_\alpha) \cup X$.
(ii) Since A is a subset of X, $A = A \cap X$ so that $A \in \mathcal{T} \Rightarrow A \in \mathcal{T}_X$. Reciprocally, if $A \in \mathcal{T}_X$, there exists $O \in \mathcal{T}$ such that $A = O \cap X$, showing that $A \in \mathcal{T}$ since both $O, X \in \mathcal{T}$.
(iii) Here, $A \subset X$ implies that $X \setminus A = X \cap (S \setminus A)$. Hence if A is closed in S, then $X \setminus A \in \mathcal{T}_X$. Reciprocally, if A is closed in X, there exists $O \in \mathcal{T}$ such that $X \setminus A = X \cap O = X \setminus (X \setminus O)$, namely $A = X \setminus O$, and further $A = (S \setminus O) \cap X$. Therefore, A is the intersection of two closed sets in S, hence it is closed itself (indeed, if C_1, C_2 are closed then $C_1 \cap C_2 = (S \setminus O_1) \cap (S \setminus O_2) = S \setminus (O_1 \cup O_2)$ is closed).

Problem 2. (i) Clearly, $M \in \mathcal{T}$. Let $x \in \cap_{j=1}^n O_j$, with $O_j \in \mathcal{T}$. There exists r_j such that $B_x(r_j) \subset O_j$ and hence $B_x(r_0) \subset \cap_{j=1}^n O_j$, where $r_0 = \min\{r_j : 1 \leq j \leq n\} > 0$ proving the finite intersection property. If now $x \in \cup_{\alpha\in I} O_\alpha$, there is α_0 such that $x \in O_{\alpha_0}$ and hence r_0 such that $B_{r_0}(x) \subset O_{\alpha_0} \subset \cup_{\alpha\in I} O_\alpha$, proving the arbitrary union property.
(ii) Let $x \in M$. Then $\{B_q(x) : q \in \mathbb{Q}\}$ is a countable set. Let N_x be a neighbourhood of x, namely $x \in N_x^o$. Since N_x^o is open, $B_r(x) \subset N_x^o \subset N_x$ for some $r > 0$ and hence all $q \in \mathbb{Q}$ with $0 < q < r$. Hence, $\{B_q(x) : q \in \mathbb{Q}\}$ is a countable neighbourhood base for x, and M is first countable since x is arbitrary.
(iii) It suffices to show that separable implies second countable. Let $D \subset M$ be a countable dense set. Then $\mathcal{B} = \{B_q(x) : q \in \mathbb{Q}, x \in D\}$ is countable base, so that M is second countable. Indeed, let O be open and $y \in O$. We show that there is $B \in \mathcal{B}$ such that $y \in B$. Clearly, $B_r(y) \subset O$ for some $r > 0$. For any $\epsilon > 0$ there is $x \in D$ such that $y \in B_\epsilon(x)$ by density. It follows that there is $0 < \delta \in \mathbb{Q}$ such that $y \in B_\delta(x) \subset B_r(y) \subset O$.
(iv) Let $x, y \in M$ be distinct. Then $3r := d(x,y) > 0$. Then $B_r(x) \cap B_r(y) = \emptyset$, proving the claim.

Problem 3. (i) $\emptyset \in \mathcal{T}$ and $S \setminus S = \emptyset$ is finite, hence $S \in \mathcal{T}$. If $Y_j \in \mathcal{T}$ for $1 \leq j \leq n$, then $S \setminus (\cap_{j=1}^n Y_j) = \cup_{j=1}^n (S \setminus Y_j)$ is finite, being a finite union of finite sets, hence $\cap_{j=1}^n Y_j \in \mathcal{T}$. Similarly, $Y_\alpha \in \mathcal{T}$ for all $\alpha \in I$, then $S \setminus (\cup_{\alpha\in I} Y_\alpha) = \cap_{\alpha\in I}(S \setminus Y_\alpha)$ is finite, being the intersection of finite sets, hence $\cup_{\alpha\in I} Y_\alpha$. Hence $\mathcal{T}$ is a topology, called the *cofinite topology*.
(ii) By definition, $S \setminus X$ and $S \setminus Y$ are finite, and hence so is their union $S \setminus (X \cap Y)$. Since S is infinite, this implies that $X \cap Y$ is not empty (in fact, it is infinite).
(iii) Let $\mathcal{N}_x = \{N_j : j \in \mathbb{N}\}$ be a countable base at x. Let $y \neq x$. Then $S \setminus \{y\}$ is an open neighbourhood of x and hence $x \in N_{j_0}^o \subset S \setminus \{y\}$ for some j_0. Hence $y \notin \cap_{j=1}^\infty N_j^o$, and hence $\cap_{j=1}^\infty N_j^o = \{x\}$. But $S \setminus \{x\} = \cup_{j=1}^\infty (S \setminus N_j^o)$ is countable, since it is a countable union of finite sets. This is in contradiction of the uncountablility of S.
(iv) Let $x \in S$ be arbitrary and let $O \in \mathcal{T}$ be such that $x \in O$. Then $S \setminus O$ is finite, and since $(x_n)_{n\in\mathbb{N}}$ does not take the same value twice, we conclude that $x_n \in O$ for all $n \geq n_x$ for some n_x. Hence $x_n \to x$ as $n \to \infty$.

Problem 4. (i) Let x be a cluster point, and let $\mathcal{N}_x$ be a countable neighbourhood base of x, such that $N_j \subset N_{j-1}$. For each j, let $x_{n_j} \in N_j$. Then $(x_{n_j})_{j\in\mathbb{N}}$ converges to x. Indeed, let M_x be a neighbourhood of x and let $N_k \subset M_x$. Then $x_{n_j} \in N_j \subset N_k$ for all $j \geq k$. Reciprocally, if $(x_{n_j})_{j\in\mathbb{N}}$ converges to x, then for any neighbourhood N_x of x, $x_{n_j} \in N_x$ for all $j \geq j_0$. Hence x is a cluster point since $\{j \geq j_0\}$ is infinite.
(ii) (a) Since $S \setminus [0,1) = \{1\}$ is finite, $[0,1)$ is open. Hence $S = [0,1) \cup \{1\}$ is smallest closed set containing $[0,1)$. (b) The set $V := S \setminus \{x_n : n \in \mathbb{N}\}$ of values of the sequence is open and $1 \in V$, so that it is a neighbourhood of 1. But $x_n \notin V$ for all $n \in \mathbb{N}$, showing that $(x_n)_{n\in\mathbb{N}}$ does not converge to 1. In particular, $([0,1), \mathcal{T})$ is not first countable.

Problem 5. (i) S is disconnected if and only if there are disjoint, open $U, V \neq \emptyset$ such that $S \setminus V = U$, which is equivalent to U being both open and closed and $U \neq S$ since $V \neq \emptyset$.
(ii) Let $\mathcal{E} = \cup_{\alpha \in I} E_\alpha$. If $\mathcal{E}$ is disconnected, there are disjoint, nonempty, open (in $\mathcal{E}$) sets U, V such that $\mathcal{E} = U \cup V$. By assumption, there exists $x \in \cap_{\alpha \in I} E_\alpha$, w.l.o.g $x \in U$. For $y \in V$, we have $y \in E_{\alpha_0}$ for some $\alpha_0 \in I$. Of course, $x \in E_{\alpha_0}$. Hence, $E_{\alpha_0} \cap U \neq \emptyset$ as well as $E_{\alpha_0} \cap V \neq \emptyset$, which is in contradiction with the fact that E_{α_0} is connected.
(iii) Assume that $\overline{X}$ is disconnected, namely $\overline{X} = U \cup V$, with U, V nonempty open and closed in $\overline{X}$. If X is connected, then either $X \cap U = \emptyset$ or $X \cap V = \emptyset$, say the second one, namely $X \subset U$. Taking the closure in $\overline{X}$ yields $\overline{X} \subset U$, namely $V = \emptyset$, which is a contradiction.
(iv) We declare $x \sim y$ if there is a connected set containing both x, y. Then $\sim$ is an equivalence relation: indeed, if $x \sim y$ and $y \sim z$, then the union of the corresponding connected sets is connected by (ii). Let C_x be the equivalence class of x, which is connected and maximal by construction. But $C_x \subset \overline{C_x}$ which is connected by (iii), and hence by maximality $C_x = \overline{C_x}$, proving that it is closed.