MATH 421/510, 2019WT2

Homework set 2 – Solution

Problem 1. (i) Pick $x \in X$ and a sequence $(x_n)_{n\in\mathbb{N}}$ converging to x. Let O be a neighbourhood of $f(x)$. Then $f^{-1}(O^o)$ is an open neighbourhood of x in X. Since $x_n \to x$, there is n_0 such that $x_n \in f^{-1}(O^o)$ for all $n \geq n_0$, namely $f(x_n) \in O^o \subset O$ for all $n \geq n_0$. Hence $f(x_n) \to f(x)$, since this holds for any neighbourhood of $f(x)$.
(ii) We first note that it suffices to prove that the preimage of any closed set is closed. Indeed if that holds, then for any open set $O \subset Y$, the set $f^{-1}(Y \setminus O)$ is closed in X, namely $X \setminus f^{-1}(Y \setminus O) = f^{-1}(O)$ is open, proving that f is continuous. Let C be closed in Y. Let $(x_n)_{n\in\mathbb{N}}$ be a convergent sequence in $f^{-1}(C)$ and let x be its limit. Then $(f(x_n))_{n\in\mathbb{N}}$ is a sequence in C which converges to $f(x)$ by assumption, and $f(x) \in C$ since C is closed, hence $x \in f^{-1}(C)$. Since X is first countable, this proves that $\overline{f^{-1}(C)} = f^{-1}(C)$, namely $f^{-1}(C)$ is closed and hence f is continuous.

Problem 2. (i) Assume that $f(S_1)$ is disconnected. Then there exist open sets $U, V \subset S_2$ such that $f(S_1) \subset U \cup V$, while $f(S_1) \cap U \neq \emptyset, f(S_1) \cap V \neq \emptyset$ and $f(S_1) \cap U \cap V = \emptyset$. By continuity, $U_1 = f^{-1}(U)$ and $V_1 = f^{-1}(V)$ are open, and their are both nonempty. Now, $f(U_1) \subset U \cap f(S_1), f(V_1) \subset V \cap f(S_1)$ are disjoint since $U \cap V = \emptyset$, hence $U_1 \cap V_1 = \emptyset$. Moreover, for any $x \in S_1$, $f(x) \in f(S_1) \subset U \cup V$, showing that $x \in U_1 \cup V_1$, namely $S_1 = U_1 \cup V_1$. Hence S_1 is disconnected, a contradiction.
(ii) Assume that S_1 is not connected, and let U, V be a separation of S_1. Let $x \in U, y \in V$. Let $f : [0, 1] \to S_1$ be any continuous function such that $f(0) = x, f(1) = y$. By (i), one of $f([0, 1]) \cap U, f([0, 1]) \cap V$ is empty, which is a contradiction. Hence S_1 is not arcwise connected.
(iii) We first prove that S is connected. Assume that it is disconnected. Then there are open sets U, V of $\mathbb{R}^2$ such that $S \subset U \cup V$, while $S \cap U \neq \emptyset, S \cap V \neq \emptyset$ and $S \cap U \cap V = \emptyset$. In particular, $(0, 0)$ belongs to only one of the the two sets $S \cap U, S \cap V$, say $(0, 0) \in U$. Let $S = S_- \cup S_+ \cup \{(0, 0)\}$, where $S_\pm$ correspond to $s \gtrless 0$. Since U is open, there is $r > 0$ such that $B_r(0) \subset U$ and hence $U \cap S_\pm \neq \emptyset$. Now, $f(s) = (s, \sin 1/s))$ is continuous from $(0, \infty)$ to $\mathbb{R}^2$ and since $(0, \infty)$ is connected, so is $S_+ = f((0, 1))$ by (i). But $S_+ \cap U \cap V \subset S \cap U \cap V = \emptyset$ and $S_+ \subset S \subset U \cup V$ imply that $S_+ \cap V = \emptyset$. Repeating the argument with $(-\infty, 0)$, we conclude that $S_- \cap V = \emptyset$. But $(0, 0) \notin V$ implies $S \cap V = (S_+ \cap V) \cup (S_- \cap V) = \emptyset$, which is a contradiction.
Reciprocally, assume that S is connected, and let $f : [0, 1] \to S$ be such that $f(0) = (0, 0), f(1) = (1, sin(1))$. Since $(0, 0)$ is closed, its preimage is closed and clearly does not contain 1. Hence, $M = \sup f^{-1}(\{(0, 0)\}) \neq 1$. Since $M \in f^{-1}(\{(0, 0)\})$, we restrict the attention to $[M, 1]$, and rescale to get a new $g : [0, 1] \to S$ such that $g(0) = (0, 0), f(1) = (1, \sin(1))$ and $g(\lambda) \neq (0, 0)$ for all $\lambda \in (0, 1]$. The second component g_2 of g is continuous and hence, for any n, there is $t_n \in (0, 1/n)$ such that $g_2(t_n) = 1$. But by continuity again, $1 = \lim_{n\to\infty} g_2(t_n) = g_2(0) = 0$, which is a contradiction.

Problem 3. (i) It is clear that $\emptyset, X \in \mathcal{T}$. Let $\{O_1, \ldots, O_N\}$ be open sets. If $\infty \notin \cap_{j=1}^N O_j$, namely $\exists j_0$ such that $\infty \notin O_{j_0}$, then $\cap_{j=1}^N O_j = S \cap (\cap_{j=1}^N O_j)$ is an open set of S since it is a finite intersection of open sets of S. But since $\infty \notin \cap_{j=1}^N O_j$, it is also an open set of X. On the other hand, assume that $\infty \in O_j$ for all $1 \leq j \leq N$. Then by definition $X \setminus O_j$ is a compact subset of S, and so is their finite union $\cup_{j=1}^N X \setminus O_j = X \setminus \cap_{j=1}^N O_j$. This shows that the intersection belongs to $\mathcal{T}$ since it contains ∞. Let now $\{O_\alpha : \alpha \in I\}$ be an arbitrary family in $\mathcal{T}$. If $\infty \notin O_\alpha$ for all $\alpha \in I$, then $\cup_{\alpha\in I} O_\alpha$ is a union of open sets of S. Since $\infty \notin \cup_{\alpha\in I} O_\alpha$ it is also open in X. If $\infty \in O_{\alpha_0}$ for some $\alpha_0 \in I$, then $X \setminus \cup_{\alpha\in I} O_\alpha$ is a closed subset of the compact $X \setminus O_{\alpha_0}$, hence compact. Since $\infty \in \cup_{\alpha\in I} O_\alpha$ it is open in X.
(ii) Since $X \setminus S = \{\infty\}$, it suffices to show that $\overline{S}$ is strictly larger than S, and equivalently that S is not closed. If it were, its complement $\{\infty\}$ would be open, namely $X \setminus \{\infty\} = S$ would be compact, a contradiction.
(iii) Let $\{O_\alpha : \alpha \in I\}$ be an open cover of X. There is an $\alpha_0 \in I$ such that $\infty \in O_{\alpha_0}$. In particular, $X \setminus O_{\alpha_0}$ is compact in S and $\{S \cap O_\alpha : \alpha \neq \alpha_0\}$ is an open cover of $X \setminus O_{\alpha_0}$. Extracting a finite cover $\{O_{\alpha_j}, 1 \leq j \leq N\}$, we conclude that $\{O_{\alpha_j}, 0 \leq j \leq N\}$ is a finite cover of X.

(iv) Let $x \neq y$ in X. If both are not ∞, then there are disjoint neighbourhoods $O_x, O_y \subset S$ of x, y that are open in S. Hence they are open in X. Let now $x = \infty, y \in S$. Since S is locally compact, there is a compact (in S) neighbourhood K_y of y. Then $O_y = N_y^0$ is open in S and hence in X and contains y. Moreover, $O_x = X \setminus K_y$ is an open (in X) and contains x, and $O_x \cap O_y = \emptyset$.
(v) Let $f : S \to \mathbb{R}$ and $y \in \mathbb{R}$. We say $\lim_{x\to\infty} f(x) = y$ if for any $\epsilon > 0$, there is a compact K_ϵ such that $|y - f(x)| < \epsilon$ for all $x \in S \setminus K_\epsilon$. Claim: f has a continuous extension to X iff $\lim_{x\to\infty} f(x)$ exists. Let us first assume that the limit exists and call that limit y. Then the function g defined by $g(x) = f(x)(x \in S)$ and $g(\infty) = y$ is a continuous extension of f. We only need to check continuity for open sets $O \ni \infty$. Then by definition of the limit, there is a compact set K_O such that $f(S \setminus K_O) \subset O$, namely $S \setminus K_O \subset f^{-1}(O)$ and $S \setminus f^{-1}(O) \subset K_O$. But $f^{-1}(O)$ being open implies that $S \setminus f^{-1}(O)$ is a closed subset of a compact set and it is therefore compact. But $g^{-1}(O) = \{\infty\} \cup f^{-1}(O)$ so that $X \setminus g^{-1}(O) = S \setminus f^{-1}(O)$ is compact, proving that $g^{-1}(O)$ is open. Reciprocally, we assume that f has a continuous extension g to X. Let $y = g(\infty)$. Then for any $\epsilon > 0$, the continuity of g implies that $g^{-1}((y-\epsilon, y+\epsilon)) = \{\infty\} \cup f^{-1}((y-\epsilon, y+\epsilon))$ is open in X, namely its complement in X, $K_\epsilon = f^{-1}((y-\epsilon, y+\epsilon))$ is compact. But that is exactly the definition of $\lim_{x\to\infty} f(x) = y$.

Problem 4. (i) Let $D = \{x \in X : f(x) = g(x)\}$. We assume that $f \neq g$ and show that D is not dense. Let x_0 be so that $f(x_0) \leq g(x_0)$. Since Y is Hausdorff, there are disjoint open sets O_f, O_g with $f(x_0) \in O_f, g(x_0) \in O_g$. By continuity, $O = f^{-1}(O_f) \cap g^{-1}(O_g)$ is open in X, and nonempty since $x_0 \in O$. Now, $f(O) \subset O_f$ and $g(O) \subset O_g$ are disjoint, namely $f(x) \neq g(x)$ for all $x \in O$. In other words $O \cap D = \emptyset$ and hence $D \subset X \setminus O$, which is closed, so that $\overline{D} \subset X \setminus O \neq X$.
(ii) Since $f^{-1}(\emptyset) = \emptyset$ and by definition $f^{-1}(Y) = Y$, both of which are open, f is continuous.
(iii) Let $f : \mathbb{R} \to Y$ be the indicator function of $\mathbb{Q}$, which is continuous when Y is equipped with the trivial topology, and let $g = 1$. Then f and g agree on the dense set $\mathbb{Q}$, but they are not equal. Of course, this is because Y is not Hausdorff.