Homework set 3 – Solution

Problem 1. (i) Let $x, x' \in X$ and $y \in E$. Taking the infimum of $d(x,y) \leq d(x,x') + d(x',y)$ over $y \in E$ implies that $d_E(x) \leq d(x,x') + d_E(x')$. Combining this inequality with the one obtained by interchanging x with x', we conclude that $|d_E(x) - d_E(x')| \leq d(x,x')$, proving the uniform continuity of d_E.
(ii) Since A is closed, $x \in A \Leftrightarrow d_A(x) = 0$. Since $\Rightarrow$ is immediate, we prove $\Leftarrow$. By definition of the infimum, $d_A(x) = 0$ implies that there is a sequence $(x_n)_{n\in\mathbb{N}}$ in A such that $d(x, x_n) \to 0$ as $n \to \infty$. Hence $x \in \overline{A} = A$. Now, since A, B are disjoint, $d_A(x) + d_B(x) > 0$ for all $x \in X$, and therefore f is continuous by (i). Moreover, $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ for all $x \in B$ and $0 \leq f \leq 1$. In other words, f provides a a slightly weaker separation of $K = B$ and $U = X \setminus A$ under weaker assumptions: B is not compact and $\mathrm{supp}(f) = \overline{X \setminus A}$.

Problem 2. (i) Let $\{O_1, \ldots, O_N\}$ be a finite open subcover of K (here we insist with the use of the relative topology on K). For any $1 \leq j \leq N$, we define $f_j : K \to [0,\infty)$ by $f_j(x) = d_{K\setminus O_j}(x)$ as in Problem 1. Each f_j is continuous and so is $f = f_1 \vee \cdots \vee f_N$. For any $x \in K$, there is $1 \leq j_0 \leq N$ such that $x \in O_{j_0}$, and hence there is δ_x such that $B_{\delta_x}(x) \cap K \subset O_{j_0}$. It follows that $f(x) \geq f_{j_0}(x) > 0$. By continuity, $f(K)$ is a compact subset of $\mathbb{R}$, and hence $2r = \min\{f(x) : x \in K\} > 0$. By the definition of f, for any $x \in K$, there is $1 \leq i_0 \leq N$ such that $f_{i_0}(x) > r$, namely $B_r(x) \cap K \subset O_{i_0}$, which is what we had set to prove.
(ii) Apply the above to the cover with one element $U \cap K$.
Remark. In other words: In a metric space, for any neighbourhood U of a compact set K, the distance between K and U^c is strictly positive.

Problem 3. We first note that

$$|x| - P_{n+1}(x) = (|x| - P_n(x))\left(1 - \frac{|x| + P_n(x)}{2}\right).$$

Assume that $|x| \leq 1$. Then $0 \leq P_0 \leq |x|$. Moreover, $0 \leq P_n(x) \leq |x|$ implies $0 \leq 1 - \frac{|x|+P_n(x)}{2} \leq 1$ and hence $0 \leq P_{n+1}(x) \leq |x|$. It follows that $0 \leq P_n(x) \leq |x|$ for all $n \in \mathbb{N}$. With this, $P_{n+1}(x) - P_n(x) = (x^2 - (P_n(x))^2)/2 \geq 0$, so that $0 \leq P_n(x) \leq P_{n+1}(x) \leq |x|$ for all $n \in \mathbb{N}$. Hence, $(P_n(x))_{n\in\mathbb{N}}$ is convergent for any $|x| \leq 1$. The limit $L(x) = \lim_{n\to\infty} P_n(x)$ satisfies $0 = x^2 - L(x)$ and hence $L(x) = |x|$ since $L(x) \geq 0$. The convergence is uniform by Dini's theorem.
Remark. With this in hand, the proof of Stone-Weierstrass does not require the classical Weirstrass result.

Problem 4. (i) If f is continuous, then for any $a \in \mathbb{R}$, both $f^{-1}((-\infty, a))$ and $f^{-1}((a,\infty))$ are open, proving that f is both l.s.c. and u.s.c. Reciprocally, for any $a < b$, $f^{-1}((a,b)) = f^{-1}((-\infty, b)) \cap f^{-1}(a,\infty))$, which is open if f is both l.s.c. and u.s.c. This proves continuity since $\{(a,b) : -\infty < a < b < \infty\}$ is a base for the metric topology on $\mathbb{R}$. Indeed: let $\mathcal{B}$ be a base for a topology and assume that $f^{-1}(B)$ is open for all $B \in \mathcal{B}$. Any open set can be written as $O = \cup_{\alpha\in I} B_\alpha$, where $B_\alpha \in \mathcal{B}$ and so $f^{-1}(O) = \{x : f(x) \in \cup_{\alpha\in I} B_\alpha\} = \cup_{\alpha\in I} f^{-1}(B_\alpha)$ is open.
(ii) If O is open, then

$$\{x \in S : \chi_O(x) > a\} = \begin{cases} \emptyset & \text{if } a \geq 1 \\ O & \text{if } 0 \leq a < 1 \\ S & \text{if } a < 0 \end{cases}$$

proving that χ_O is l.s.c. since all three $\emptyset, O, S$ are open.
(iii) Let C be closed, namely $C = O^c$ where O is open. Then $\chi_C(x) = 1 - \chi_O(x)$ proving that χ_C is u.s.c by (ii). Indeed, if f is l.s.c then $-f$ is u.s.c. since $\{x : f(x) > a\} = \{x : -f(x) < -a\}$.

(iv) It suffices to note that $\{x \in S : \sup\{f_\alpha(x) : \alpha \in I\} > a\} = \cup_{\alpha\in I}\{x : f_\alpha(x) > a\}$. Hence it is open if all f_α are l.s.c.

Problem 5. (i) Let $\{O_\alpha : \alpha \in I\}$ be an open cover of $X \times Y$. For any $(x, y) \in X \times Y$, there is an $\alpha(x, y)$ such that $(x, y) \in O_{\alpha(x,y)}$. Since simple products of open sets form a base, there are $U_{(x,y)} \in \mathcal{T}_X, V_{(x,y)} \in \mathcal{T}_Y$ such that $(x, y) \in U_{(x,y)} \times V_{(x,y)} \subset O_{\alpha(x,y)}$. For any fixed $x \in X$, the collection $\{V_{(x,y)} : y \in Y\}$ is an open cover of Y, from which we extract a finite subcover indexed by $\{y_{x,1}, \dots, y_{x,n}\}$. The set $U_x = \cap_{j=1}^n U_{(x,y_{x,j})}$ is open and contains x. Hence, the collection $\{U_x : x \in X\}$ is an open cover of X, from which we extract a finite subcover indexed by $\{x_1, \dots, x_m\}$. It follows that $\{O_{\alpha(x_i,y_{x_i,j})} : 1 \le i \le m, 1 \le j \le n\}$ is a finite subcover of $X \times Y$.

(ii) Let $\mathcal{A} = \{\sum_{j=1}^n g_j(x)h_j(y) : n \in \mathbb{N}$ and $g_j \in C_\mathbb{R}(X), h_j \in C_\mathbb{R}(Y)$ for all $1 \le j \le n\}$. Clearly, $\mathcal{A}$ is an algebra. $1 \in \mathcal{A}$ since 1 corresponds to $n = 1, g_1 = h_1 = 1$. Let $(x, y) \neq (x', y')$, without loss $x \neq x'$. By Urysohn's lemma applied to $K = \{x\}$ and $U = X \setminus \{x'\}$, there is a function $g \in C_\mathbb{R}(X)$ such that $g(x) = 1$ and $g(x') = 0$. g is identified with a function in $\mathcal{A}$ by setting $n = 1, h = 1$ and hence $\mathcal{A}$ separates points. By (i), $X \times Y$ is compact so that $\mathcal{A}$ is dense in $C_\mathbb{R}(X, Y)$ by Stone-Weierstrass, concluding the proof.