MATH 421/510, 2019WT2

Homework set 5 – Solution

Problem 1. Clearly, C, C_0 are vector spaces. So we check completeness. Let $(z^n)_{n\in\mathbb{N}}$ be a Cauchy sequence in C, and let $\lim_{j\to\infty} z_j^n = Z^n$ for any $n \in \mathbb{N}$. First of all,

$$|Z^n - Z^m| = |\lim_{j\to\infty}(z_j^n - z_j^m)| \le \sup\{|z_j^n - z_j^m| : j \in \mathbb{N}\} = \|(z^n) - (z^m)\|,$$

so that $(Z^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}$ and hence convergent. Let $Z = \lim_{n\to\infty} Z^n$. Secondly, $|z_j^n - z_j^m| \le \sup\{|z_j^n - z_j^m| : j \in \mathbb{N}\} = \|(z^n) - (z^m)\|$ so that for any fixed $j \in \mathbb{N}$ the sequence $(z_j^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}$ and hence convergent. Let $w_j = \lim_{n\to\infty} z_j^n$. We now claim (a) that $(w_j)_{j\in\mathbb{N}}$ is a convergent sequence, namely $(w_j)_{j\in\mathbb{N}} \in C$ with $\lim_{j\to\infty} w_j = Z$, and (b) that $\lim_{n\to\infty}(z^n) = (w_j)_{j\in\mathbb{N}}$. (a,b) together conclude the proof in the case C. The case C_0 follows then by imposing $Z^n = 0$ for all n and hence $Z = 0$.
Proof of (a). Let $\epsilon > 0$. There is $m \in \mathbb{N}$ such that $|z_j^n - z_j^m| < \epsilon/3$ for all $n > m$ and $j \in \mathbb{N}$ as well as $|Z^m - Z| < \epsilon/3$. Then for all $j \in \mathbb{N}$,

$$|w_j - z_j^m| = \lim_{n\to\infty} |z_j^n - z_j^m| \le \frac{\epsilon}{3}.$$

Let now N be so that $|z_j^m - Z^m| < \epsilon/3$ for all $j > N$. Then

$$|w_j - Z| \le |w_j - z_j^m| + |z_j^m - Z^m| + |Z^m - Z| < \epsilon$$

for all $j > N$ indeed.
Proof of (b). Let $\epsilon > 0$. There is $N \in \mathbb{N}$ such that $\|z^n - z^m\| < \epsilon$ for all $n, m \ge N$. But then

$$\|z^n - w\| = \sup\{|z_j^n - w_j| : j \in \mathbb{N}\} = \sup\{\lim_{m\to\infty} |z_j^n - z_j^m| : j \in \mathbb{N}\} \le \lim_{m\to\infty} \|z^n - z^m\| < \epsilon$$

for all $n \ge N$.

Problem 2. (i) We first note the following, which is simply a rephrasing of the definition of the essential supremum. For any $\epsilon > 0$, we have that $\|g\|_\infty + \epsilon \in \{M : |g(x)| \le M$ for μ-almost every $x \in \Omega\}$, namely there is a set E_ϵ of measure zero such that

$$\sup\{|g(x)| : x \in \Omega \setminus E_\epsilon\} \le \|g\|_\infty + \epsilon.$$

Assume that $(f_n)_{n\in\mathbb{N}}$ converges to f in the $\|\cdot\|_\infty$-norm. For each $n \in \mathbb{N}$ there is a set $E_n \subset \Omega$ of measure zero such that

$$\sup\{|f_n(x) - f(x)| : x \in \Omega \setminus E_n\} \le \|f_n - f\|_\infty + \frac{1}{n}. \tag{1}$$

The set $E = \bigcup_{n\in\mathbb{N}} E_n$ is a countable union of sets of measure zero, hence it is itself a set of measure zero. Furthermore,

$$\limsup_{n\to\infty} \sup\{|f_n(x) - f(x)| : x \in \Omega \setminus E\} \le \limsup_{n\to\infty} \sup\{|f_n(x) - f(x)| : x \in \Omega \setminus E_n\} = 0$$

by (1). Thus $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f on $\Omega \setminus E$.
Reciprocally, assume that there is a set $E \subset \Omega$ of measure zero such that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f on $\Omega \setminus E$. As

$$\limsup_{n\to\infty} \|f_n - f\|_\infty \le \limsup_{n\to\infty} \sup\{|f_n(x) - f(x)| : x \in \Omega \setminus E\} = 0,$$

f_n converges to f in the $\|\cdot\|_\infty$-norm.

(ii) Let $\epsilon > 0$. Let $\delta > 0$. Then for any $x \in \Omega$, there is $N(\delta, x)$ such that $|f_n(x) - f(x)| < \delta$ for $n \geq N(\delta, x)$. For $N \in \mathbb{N}$, the sets $S(\delta, N) = \{x \in \Omega : M(\delta, x) \leq N\}$ for a non-decreasing sequence in both δ and N, and let $S(\delta) = \cup_{N\in\mathbb{N}} S(\delta, N)$. By assumption, almost every $x \in \Omega$ belongs to some $S(\delta, N)$, we have that $\mu(S(\delta)) = \lim_{N\to\infty} \mu(S(\delta, N)) = \mu(\Omega)$. In particular, for any $\rho > 0$, $\mu(S(\delta, N)) \geq \mu(\Omega) - \rho$ for N large enough. Let now $(\delta_j)_{j\in\mathbb{N}}$ be a sequence of positive numbers tending to zero and let $(N_j)_{j\in\mathbb{N}}$ be so that $\mu(S(\delta_j, N_j)) \geq \mu(\Omega) - 2^{-j}\epsilon$. By construction, the set $R_\epsilon = \cap_{j\in\mathbb{N}} S(\delta_j, N_j)$ is so that $f_n \to f$ uniformly on R_ϵ. Moreover,

$$\mu(R_\epsilon) = \mu(\cup_{j\in\mathbb{N}} \Omega \setminus S(\delta_j, N_j)) \leq \sum_{j=1}^{\infty} 2^{-j}\epsilon = \epsilon$$

so that R_ϵ satisfies the claim. *Remark.* This is known as *Egorov's theorem*

Problem 3. (i) The substitution $x \to z = x/y$ and the scaling property of the kernel K (by y) yield $\int_0^\infty |K(x,y)f(x)|dx = \int_0^\infty |K(z,1)f_z(y)|dz$. Here $f_z(y) = f(zy)$ for which $\|f_z\|_p = z^{-1/p}\|f\|_p$ by scaling. In particular $y \mapsto |K(z,1)f_z(y)|$ is in L^p and $\int_0^\infty |K(z,1)|\|f_z\|_p dz = C\|f\|_p < \infty$ by the integrability assumption on K. The claim now follows from the generalized Minkowski's inequality, namely $\|Tf\|_p \leq \int_0^\infty |K(z,1)|\|f_z\|_p dz = C\|f\|_p$.
(ii) The inequality is $\|Tf\|_p^p \leq C^p\|f\|_p^p$ of (i) with the choices

$$f(x) = \frac{h(x)}{x^{(1+r-p)/p}} \qquad K(x,y) = \chi_{\{0<x<y\}}(x,y)\frac{1}{y^{(1+r)/p}}x^{(1+r-p)/p}.$$

On the one hand, this choice gives $(Tf)(y) = y^{-(1+r)/p}\int_0^y h(x)dx$ and $\|Tf\|_p^p$ is the left hand side of the inequality. On the other hand, $\|f\|_p^p$ is the integral on the right hand side of the inequality. We compute the constant as $C^p = (\int_0^1 x^{(1+r-p)/p}x^{-1/p}dx)^p = (p/r)^p$ indeed.
Remark. This is called Hardy's inequality. It is often stated in the case $r = p-1$ and expressed in differential terms:

$$\int_0^\infty \left(\frac{g(y)}{y}\right)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty (g'(x))^p\, dx$$

Problem 4. We establish differentiability at $t = 0$. Differentiability for any t follows from the argument below upon replacing f with $f + tg$. For any $z, w \in \mathbb{C}$,

$$\lim_{t\to 0} t^{-1}|z + tw|^p = \frac{d}{dt}(z + tw)^{p/2}(\bar{z} + t\bar{w})^{p/2}|_{t=0} = \frac{p}{2}|z|^{p-2}(z\bar{w} + \bar{z}w)$$

which reduces the proof to the exchange of differentiation and integration. The convexity of $x \mapsto |x|^p$ for $p \geq 1$ yields

$$|f + tg|^p \leq (1-t)|f|^p + t|f+g|^p \qquad (0 \leq t \leq 1)$$
$$|f + tg|^p \leq (1+t)|f|^p - t|f-g|^p \qquad (-1 \leq t \leq 0)$$

as well as

$$|f + tw|^p \geq |f|^p + t(p/2)|f|^{p-2}(f\bar{g} + \bar{f}g).$$

Hence,

$$(p/2)|f|^{p-2}(f\bar{g} + \bar{f}g) \leq \frac{1}{t}\left(|f(x) + tg(x)|^p - |f(x)|^p\right) \leq |f(x) + g(x)|^p - |f(x)|^p \qquad (0 < t \leq 1)$$
$$|f(x)|^p - |f(x) - g(x)|^p \leq \frac{1}{t}\left(|f(x) + tg(x)|^p - |f(x)|^p\right) \leq (p/2)|f|^{p-2}(f\bar{g} + \bar{f}g) \qquad (-1 \leq t < 0)$$

which implies that the limit can be interchanged with the integral by dominated convergence. Indeed, $|f|^p$ and $|f \pm g|^p$ are integrable, and so is $|f|^{p-2}(f\bar{g} + \bar{f}g)$ by Hölder's inequality:

$$\left|\int_\Omega |f|^{p-2}(f\bar{g} + \bar{f}g)d\mu\right| \leq 2\|f\|_p\|g\|_p$$